2.

$$x(t) = c_1 \begin{pmatrix} -5\cos t \\ 2\cos t - \sin t \end{pmatrix} + c_2 \begin{pmatrix} -5\sin t \\ 2\sin t + \cos t \end{pmatrix}$$

6. $$\begin{bmatrix} \cos 2t & \sin 2t \\ \sin 2t - \cos 2t & -\sin 2t - \cos 2t \end{bmatrix}$$

13. $A = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}$

$$0 = |A - rI| = \begin{vmatrix} -3-r & -1 \\ 2 & -1-r \end{vmatrix} = (-3-r)(-1-r) + 1 \cdot 2$$

$$= r^2 + 4r + 3 + 2 = r^2 + 4r + 5$$

$$= (r+2)^2 + 1$$

$$\therefore r = -2 \pm i$$

$r_1 = -2 + i$: $0 = (A - r_1 I)\vec{u} = \begin{bmatrix} -1-i & -1 \\ 2 & 1-i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{cases} -(1+i)a - b = 0 & \rightarrow -(1-i)(1+i)a - (1-i)b = 0 \\ 2a + (1-i)b = 0 & \Leftrightarrow \therefore -2a - (1-i)b = 0 \end{cases}$$

$$\therefore 2a = -(1-i)b$$

If $b = 2$, $a = -1 + i$

$$\therefore \vec{u}_1 = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$$

$$e^{r_1 t}\vec{u}_1 = e^{-2t}(\cos t + i\sin t)\begin{bmatrix} -1+i \\ 2 \end{bmatrix}$$

$$= e^{-2t}\begin{bmatrix} -\cos t - \sin t + i(\cos t - \sin t) \\ 2\cos t + i\, 2\sin t \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-2t}(\cos t + \sin t) \\ 2e^{-2t}\cos t \end{bmatrix} + i \begin{bmatrix} e^{-2t}(\cos t - \sin t) \\ 2e^{-2t}\sin t \end{bmatrix}$$

$$\therefore \vec{x}(t) = c_1 e^{-2t}\begin{bmatrix} -\cos t - \sin t \\ 2\cos t \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} \cos t - \sin t \\ 2\sin t \end{bmatrix}$$

(a) $\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\therefore -c_1 + c_2 = -1 \quad \rightarrow \quad c_2 = -1$$
$$2c_1 = 0 \quad \rightarrow \quad c_1 = 0$$

$$\therefore \vec{x}(t) = -e^{-2t}\begin{bmatrix} \cos t - \sin t \\ 2\sin t \end{bmatrix} = \begin{bmatrix} e^{-2t}(\sin t - \cos t) \\ -2e^{-2t}\sin t \end{bmatrix}$$

(b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x(\pi) = c_1 e^{-2\pi}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-2\pi}\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\therefore (c_1 e^{-2\pi}) - (c_2 e^{-2\pi}) = 1 \quad \rightarrow \quad c_2 e^{-2\pi} = -\tfrac{1}{2}$$
$$(c_1 e^{-2\pi})(-2) = -1 \quad \rightarrow \quad c_1 e^{-2\pi} = \tfrac{1}{2}$$

$$\therefore c_1 = \tfrac{1}{2}e^{2\pi}, \quad c_2 = -\tfrac{1}{2}e^{2\pi}$$

$$\therefore \vec{x}(t) = \tfrac{1}{2}e^{2\pi}e^{-2t}\begin{bmatrix} -\cos t - \sin t \\ 2\cos t \end{bmatrix} - \tfrac{1}{2}e^{2\pi}e^{-2t}\begin{bmatrix} \cos t - \sin t \\ 2\sin t \end{bmatrix}$$

$$= e^{-2(t-\pi)}\begin{bmatrix} -\tfrac{1}{2}\cos t \cancel{- \tfrac{1}{2}\sin t} - \tfrac{1}{2}\cos t \cancel{+ \tfrac{1}{2}\sin t} \\ \cos t - \sin t \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-2(t-\pi)}\cos t \\ e^{-2(t-\pi)}(\cos t - \sin t) \end{bmatrix}$$