

Math- 440: Handout1, Monday March 28, 2005

Math 340 - Review

(A1) Sampling without Replacement

We select a card from a deck of n different cards then we remove it from the deck. Next, we select another card from the $(n - 1)$ remaining cards and remove this one too. Clearly, there are $n \times (n - 1)$ ways to do this. In general, if we repeat this process r times, the possible number of outcomes will be:

$$n \times (n - 1) \times (n - 2) \times \dots \times (n - r + 1).$$

This in fact, is what we call $P_{n,r}$.

It becomes trivial that: $P_{n,n} = n!$

Note that $P_{n,r} \times (n - r)! = n \times (n - 1) \times (n - 2) \times \dots \times (n - r + 1) \times (n - r)! = n!$

Hence,

$$P_{n,r} = \frac{n!}{(n - r)!}$$

- (B1) In Maryland's lottery, players pick six different integers between 1 and 49, order of selection is irrelevant. Six numbers among 49 are randomly selected as winning numbers. A player hits the jackpot if she/he matches all six numbers. The second big prize is rewarded to person(s) matching five numbers and the third prize goes to person(s) matching 4. Find the probabilities that: (a) Sam's ticket wins the jackpot. (b) Sam's ticket wins the second prize. (c) Sam's ticket wins the third prize.
- (B3) From an ordinary deck of 52 cards, seven are drawn at random and without replacement. What is the probability that at least one of the cards is a king?
- (B4) What is the probability that a poker hand is a full house? In the game of poker, a hand of 5 randomly selected cards is called full if there are three cards are from one denomination and the other 2 cards are from another denomination. For example, a hand of three kings and two 3s.
- (B5) The mathematics department consists of 25 full professors, 15 associate professors, and 35 assistant professors. A committee of 6 is selected at random from each faculty of the department. (a) Find the probability that all members of the committee are assistant professors. (b) What is the probability that the committee of 6 is composed of 2 full professors, 3 associate professors and 1 assistant professor?
- (B6) In a hand of 13 cards chosen from an ordinary deck of 52, find the probability that the hand is composed of exactly 3 clubs, 4 diamonds, 4 hearts and 2 spades. Remember:

$$P(A) = \sum_{j=1}^k P(B_j)P(A|B_j) \quad (1)$$

$$P(A|C) = \sum_{j=1}^k P(B_j|C)P(A|B_j \cap C) \quad (2)$$

- (C1) Two boxes contain long bolts and short bolts. Suppose that one box contains 60 long bolts and 40 short bolts, and the other box contains 10 long bolts and 20 short bolts. Suppose also that one box is selected at random and a bolt is then selected at random from that box. What is the probability that this bolt is long?
- (C2) (Mood, Graybill and Boes, 1974) There are five urns, and they are numbered 1 to 5. Each urn contains 10 balls. Urn i has i defective balls and $10 - i$ non-defective balls, for $i = 1, \dots, 5$. For example, urn 3 has 3 defective balls and 7 non-defective balls. Consider the following random experiment:
First, an urn is selected at random, and then a ball is selected at random from the selected urn. (The experimenter does not know which urn is selected.) We are interested in two questions:
(1) What is the probability that a defective ball is selected?
(2) If we have already selected a ball and noted that it is defective, what is the probability that it came from urn 5?
- (C3) (DeGroot, Schervish, 2002) Three machines M_1 , M_2 , and M_3 were used for producing a large batch of similar manufactured items. Suppose that 20 percent of the items were produced by machine M_1 , 30 percent of the items were produced by machine M_2 , and 50 percent by machine M_3 . Suppose further that 1 percent of the items produced by machine M_1 are defective, that 2 percent produced by machine M_2 are defective, and 3 percent produced by machine M_3 are defective. Finally, suppose that one item is selected at random from the entire batch, and it is found to be defective. Find the probability that this item was produced by machine M_2 .
- (D1) Let's assume that the *cdf* of a random variable x is as follows:
- $$F(x) = \begin{cases} 0 & x < 2 \\ (x - 2)^4/16 & 2 \leq x < 4 \\ 1 & \text{otherwise} \end{cases}$$
- (a) Graph $F(x)$.
- (b) Calculate the following probabilities:
 $P(X \leq 3)$, $P(X < 3)$, $P(X > 2.5)$, $P(1.5 < X \leq 3.4)$.
- (D2) Suppose that a bus arrives at the station every day between 10:00 A.M. and 10:30 A.M., at random. Let X be the arrival time; find the *cdf* of X and sketch its graph.

- (D3) While walking in a certain park, the time X , in minutes, between seeing two people smoking has a probability density function (*pdf*) of the following form:

$$f(x) = \begin{cases} \lambda x e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Calculate the value of λ .
 (b) Find the *cdf* of X .
 (c) What is the probability that Sam, who has just seen a person smoking, will see another person smoking in 2 to 5 minutes? In at least 7 minutes?

- (E1) Let λ be a positive number. Let, $F(x, y) = \begin{cases} 1 - \lambda e^{-\lambda(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$
 Determine whether F could be a joint-*pdf* of the two variables X and Y ?

- (E2) Let X and Y be two *independent* random variables. Let $g : \mathcal{R} \rightarrow \mathcal{R}$ and $h : \mathcal{R} \rightarrow \mathcal{R}$ be two real-values functions. Prove that $g(X)$ and $h(Y)$ are also independent. That is, prove the following statement:

$$P(g(X) \leq a, h(Y) \leq b) = P(g(X) \leq a) \times P(h(Y) \leq b)$$

for any two real numbers a and b .

- (E3) Stores A and B , which belong to the same owner, are located in two different towns. If the probability density function of the weekly profit of each store, in thousands of Dollars, is given by:

$$f(x) = \begin{cases} x/4 & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

and the profit of one store is independent of the other, what is the probability that next week one store makes at least \$ 500 more than the other one?

- (E4) Let the joint *pmf* of X and Y be given by:

$$f(x, y) = \begin{cases} \frac{1}{15}(x + y) & x = 0, 1, 2, y = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $f(x|y)$ and $P(X = 0|Y = 2)$.

- (E5) Let the random variable X follow a uniform distribution on the interval $(0, 1)$. That

$$\text{is, } f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Also, let $f(y|x)$ follow a Binomial distribution with n (the number of experiments) and x as the probability of success! That is:

$$g(y|x) = \begin{cases} \binom{n}{y} x^y (1-x)^{n-y} & y = 0, 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

Find $g_2(x|y)$ or the *pdf* of $X|y$.

Note: This is very close to what Thomas Bayes did in his hallmark paper: *Essay towards solving a problem in the doctrine of chances* which was appeared at *Philosophical Transactions of the Royal Society of London* in 1764.

(F1) Prove that the following relation holds for any continuous random variable such as X :

$$E(X) = \int_0^{\infty} [1 - F(t)]dt - \int_0^{\infty} F(-t)dt.$$

Note that, a direct consequence of the above relation is the following:

$$E(X) = \int_0^{\infty} P(X > t)dt - \int_0^{\infty} P(X \leq -t)dt.$$

(F2) The time elapsed (in minutes), between the placement of an order of pizza and its delivery is random with density function: $f(x) = \left\{ \begin{array}{ll} \frac{1}{15} & 25 < x < 40 \\ 0 & \text{otherwise} \end{array} \right\}$

(a) Obtain the mean and the standard deviation of X .

(b) Suppose that it takes 12 minutes for the pizza shop to bake pizza. Find the mean and the standard deviation of the time it takes for the delivery person to deliver pizza.

(G1) Find the moment generating function for a random variable that follows exponential distribution with parameter $\lambda > 0$. Find $E(X)$ and $Var(X)$. (note that the random variable X is not bounded from the above.)

(G2) Find the moment generating function for a random variable that follows Poisson distribution with parameter $\lambda > 0$.

(G3) Let X and Y have a continuous distribution with joint *pdf*:

$$f(x, y) = \left\{ \begin{array}{ll} x + y & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{array} \right\}$$

Find $Cov(X, Y)$.

(G4) Prove *Shwarz inequality*:

$$[E(UV)]^2 \leq E(U^2)E(V^2)$$

hint: See page 216.

(H1) **Convergence in Probability** Let X_1, X_2, \dots, X_n be a sequence of random variables. This sequence converges to a given number b if the probability distribution of X_n becomes more and more concentrated around b as $n \rightarrow \infty$.

Formally, this says: The sequence converges to b in probability if $\forall \epsilon > 0$:

$$\lim_{n \rightarrow \infty} Pr(|X_n - b| < \epsilon) = 1 \quad (3)$$

We write this as $X_n \xrightarrow{P} b$

(H2) **Law of Large Numbers.** Let X_1, \dots, X_n form a random sample from a distribution for which the mean is μ and the variance exists. Let \bar{X}_n represent the sample mean. Then:

$$\bar{X}_n \xrightarrow{P} \mu. \quad (4)$$

(H3) Consider flipping a coin for which the probability of heads is p . Let X_i denote the outcome of a single toss (0 or 1). Hence:

$$E(X_i) = Pr(X_i = 1) = p.$$

Note that, the fraction of heads after n tosses is \bar{X}_n . According to the law of large numbers, \bar{X}_n converges in probability to p . This does not mean that \bar{X}_n will numerically equal p . It means that, **when n is large, the distribution of \bar{X}_n is tightly concentrated around p** . Now, suppose $p = \frac{1}{2}$. How large n should be so that $Pr(0.4 \leq \bar{X}_n \leq 0.6) \geq 0.7$?

Note that: $E(\bar{X}_n) = 0.5$, and $Var(\bar{X}_n) = \frac{Var(X_i)}{n} = \frac{p(1-p)}{n} = \frac{1/2 \times 1/2}{n} = \frac{1}{4n}$. Hence, by Chebyshev's inequality:

$$Pr(0.4 \leq \bar{X}_n \leq 0.6) = Pr(|\bar{X}_n - 0.5| \leq 0.1) = Pr(|\bar{X}_n - \mu| \leq 0.1) = 1 - Pr(|\bar{X}_n - \mu| > 0.1) \geq 1 - \frac{1}{4n \times (0.1)^2} = 1 - \frac{25}{n}.$$

We want this to be larger than 0.7. In other words, $1 - \frac{25}{n} \geq 0.7$. This is accomplished when $n \geq 84$.

(I1) Here are a couple of examples for Poisson's approximation to Binomial :

- (a) Let X be the number of winning tickets among the California lottery tickets sold in Bakersfield during one week. Then calling winning ticket successes, we have that X is a binomial random variable. Since n , the total number of tickets sold in Bakersfield, is large, p the probability of winning is small, and the average number of tickets sold is large, then X is approximately a Poisson random variable.
- (b) Let X be the number of spikes or firing activities of a particular neuronal cell in a 500 millisecond span of time. Then calling the spike occurrence success, we have that X is a binomial random variable. Since n , the total number of spike occurrences is very large, and p the probability of a spike activity in a one millisecond unit of time is relatively small, then X the number of spike occurrences follows a Poisson random variable.

(I2) Every week the average number of wrong-number phone calls received by a certain mail-order house is seven. What is the probability that they will receive:

- (a) Two wrong calls tomorrow.
- (b) At least one wrong call tomorrow?

(I3) The atoms of a radioactive element are randomly disintegrating. If every gram of this element, on average, emits 3.9 alpha particles per second, what is the probability that during the next second the number of alpha particles emitted from 1 gram is:

- (a) At most 6.
- (b) At least 2.
- (c) At least 3 and at most 6.

(J1) Suppose that the joint *pmf* of X and Y is given by:

$$f(x, y) = \left\{ \begin{array}{ll} 1/7 & x = 5, y = 0 \\ 1/7 & x = 5, y = 3 \\ 1/7 & x = 5, y = 4 \\ 3/7 & x = 8, y = 0 \\ 1/7 & x = 8, y = 4 \\ 0 & \text{otherwise} \end{array} \right\}$$

- Find $E(X|Y = 0)$
- Find $E(X|Y = 4)$
- Find $E(X|Y = 3)$

(J2) Let X and Y have joint density:

$$f(x, y) = \left\{ \begin{array}{ll} 4x^2y + 2y^5 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{array} \right\}$$

Find $E(X|Y = y)$.

(K1) Consider the time T between event occurrences in a Poisson process. Suppose that the time interval $[0, t]$ is divided into n subintervals of length $\delta = \frac{t}{n}$. The probability that the inter-event time T exceeds t seconds is equivalent to no event occurring in t seconds (or in Bernoulli trials):

$$Pr[T > t] = Pr(\text{no events in } t \text{ seconds}) = (1 - p)^n = \left(1 - \frac{\lambda t}{n}\right)^n \rightarrow e^{-\lambda t}$$

as $n \rightarrow \infty$. This means that $Pr(T \leq t) = 1 - e^{-\lambda t}$ which is the CDF of an exponential distribution with the rate $\frac{1}{\lambda}$. To summarize:

The inter-event times in a Poisson process form an iid sequence of random variables with mean $\frac{1}{\lambda}$.