Some open problems in topological groups

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Disclaimer

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- This is NOT a power point presentation.
Tychonoff, $\mathbb{T}, G$

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- $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. As a model for $\mathbb{T}$ we take

$$\left(\left(-\frac{1}{2}, \frac{1}{2}\right], + \mod 1\right)$$
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- $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. As a model for $\mathbb{T}$ we take
  
  $\left( \left[ -\frac{1}{2}, \frac{1}{2} \right], + \text{ mod } 1 \right)$

- $G = (\Gamma, \tau)$ is an Abelian topological group with underlying group $\Gamma$ and group topology $\tau$. 
Character group

- The character group of $G$ is given by

\[
\hat{G} := \{ f : G \rightarrow \mathbb{T} : f \ \tau\text{-continuous homomorphism}\}
\]

Elements of $\hat{G}$ are called characters.
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Elements of $\hat{G}$ are called characters.

• We assume $G$ to be maximally almost periodic (MAP) i.e.,

$$\forall g \in G \setminus \{0\} \exists f \in \hat{G} [f(g) \neq 0]$$
Bohr topology

• For $G$ as above, we define

$$\tau_w := \text{weakest topology that makes the elements of } \hat{G} \text{ continuous}$$

$$G^+ := (\Gamma, \tau_w)$$
Totally bounded groups

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• (Weil, 1937) \( G \) is totally bounded if and only if it is a dense subgroup of a compact group, which we denote by \( \overline{G} \).

• (Comfort-Ross, 1964) \( G = (\Gamma, \tau) \) is totally bounded if and only if \( \tau = \tau_w \).
Bohr compactification

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- As groups, $\hat{G} = \hat{bG}$. 
Leptin, Glicksberg

- (Leptin, 1954) *For G discrete* ($\tau = \mathcal{P}(\Gamma)$) and $K \subseteq G$:

$$K \text{ finite } \iff K \text{ compact in } G^+$$
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- (Glicksberg, 1962) For $G$ locally compact

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- (Glicksberg, 1962) *For G locally compact*

  $$K \text{ compact in } G \iff K \text{ compact in } G^+$$

- Many authors have reproven Glicksberg’s theorem.
For $G$ locally compact, let $N$ be a metrizable subgroup of $bG$ (with $N \cap G^+ = \{0\}$). If $K \subseteq G$, then

$$K \text{ compact in } G \iff K \text{ compact in } bG/N$$

$$G \rightarrow G^+ \subseteq bG \rightarrow bG/N$$
Comfort, T-A, and Wu, 1993

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- Condition may hold even for $N$ non-metrizable.
Compact-open topology on $\hat{G}$

- If $K \subset G$ is compact and $\varepsilon > 0$, a typical neighborhood of the trivial character is

$$(K, \varepsilon) := \{f \in \hat{G} : |f(k)| < \varepsilon \ \forall k \in K\}$$
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- If $G$ is discrete, then $\hat{G}$ is compact.

- If $G$ is compact, then $\hat{G}$ is discrete.
Annihilator

• If $H$ is a closed subgroup of $G$ its *Annihilator* is defined by

$$A(\hat{G}, H) := \{ f \in \hat{G} : f(h) = 0 \ \forall h \in H \}$$
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• $A(\hat{G}, H)$ is a subgroup of $\hat{G}$.

• As groups, $\hat{H}$ and $\hat{G}/A(\hat{G}, H)$ are isomorphic.
Annihilator

- If $H$ is a closed subgroup of $G$ its **Annihilator** is defined by

\[ \mathbb{A}(\hat{G}, H) := \{ f \in \hat{G} : f(h) = 0 \ \forall h \in H \} \]

- $\mathbb{A}(\hat{G}, H)$ is a subgroup of $\hat{G}$.

- As groups, $\hat{H}$ and $\hat{G}/\mathbb{A}(\hat{G}, H)$ are isomorphic.

- Hence $N$ compact and metrizable $\implies \hat{N} \simeq \hat{G}/\mathbb{A}(b\hat{G}, N)$ is countable.
\( \mathbb{A}(\hat{b}G, N) \) for \( N \) metrizable

- Hence \( \mathbb{A}(\hat{b}G, N) \) is a subgroup of \( \hat{b}G \simeq \hat{G} \) of countable index.
$A(\hat{bG}, N)$ for $N$ metrizable

- Hence $A(\hat{bG}, N)$ is a subgroup of $\hat{bG} \simeq \hat{G}$ of countable index.
- If $G$ is discrete, then $\hat{G}$ is compact.
A(\hat{bG}, N) for N metrizable

- Hence A(\hat{bG}, N) is a subgroup of \hat{bG} \simeq \hat{G} of countable index.
- If G is discrete, then \hat{G} is compact.
- As groups, \hat{G}/H and A(\hat{G}, H) are isomorphic.
\( \mathbb{A}(\hat{bG}, N) \) for \( N \) metrizable

- Hence \( \mathbb{A}(\hat{bG}, N) \) is a subgroup of \( \hat{bG} \cong \hat{G} \) of countable index.
- If \( G \) is discrete, then \( \hat{G} \) is compact.
- As groups, \( \hat{G}/H \) and \( \mathbb{A}(\hat{G}, H) \) are isomorphic.
- Therefore \( \hat{bG}/N \cong \mathbb{A}(\hat{bG}, N) \) is a (Haar) non-measurable subgroup of \( \hat{G} \).
Hence, in CTW [1993]

- For $G$ discrete, let $N$ be a metrizable subgroup of $bG$ (with $N \cap G^+ = \{0\}$). If $K \subseteq G$, then

$$K \text{ finite } \iff K \text{ compact in } bG/N$$

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  \[ K \text{ finite } \iff K \text{ compact in } bG/N \]

  \[ G \longrightarrow G^+ \subseteq bG \longrightarrow bG/N \]

- $\widehat{bG/N} \simeq \Delta(\widehat{bG}, N)$ is a (Haar) non-measurable subgroup of $\widehat{G}$. 

Sequences

• If $K$ is compact and countable, then a non-trivial sequence in $K$ converges.
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• (CTW, 1993) If $\langle x_n \rangle$ is a non-trivial sequence of elements in (discrete) $G$ and

$$A := \{ f \in \hat{G} : f(x_n) \to 0 \}$$

Then $A$ is a Borel subgroup of measure 0.
A la Leptin

- (CTW, 1993) Let $G$ be discrete and countable. If $N$ is a subgroup of $bG$ with non-measurable (in $\hat{G}$) annihilator $\mathbb{A}(\hat{bG}, N)$, then for $K \subseteq G$

$$K \text{ finite } \iff K \text{ compact in } bG/N$$
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- (CTW, 1993) Let $G$ be discrete and countable. If $N$ is a subgroup of $bG$ with non-measurable (in $\hat{G}$) annihilator $A(\hat{bG}, N)$, then for $K \subseteq G$

$$K \text{ finite } \iff K \text{ compact in } bG/N$$

- Again, $\hat{bG}/N = A(\hat{bG}, N) \subseteq \hat{bG} \simeq \hat{G}$. 
Kakutani (1943)

\[
\text{Hom}(\Gamma, \mathbb{T}) := \{ f : \Gamma \rightarrow \mathbb{T} : f \text{ group homomorphism} \}
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\[ \text{Hom}(\Gamma, \mathbb{T}) := \{ f : \Gamma \rightarrow \mathbb{T} : f \text{ group homomorphism}\} \]

- \text{Hom}(\Gamma, \mathbb{T}) \text{ has } 2^{2|\Gamma|} \text{-many subgroups } A_\zeta \text{ which separate points of } \Gamma \text{ i.e.,}\]

\[ \forall g \in \Gamma \setminus \{0\} \exists f \in A_\zeta [f(g) \neq 0]. \]
Duality, CR (1964)

- If $G = \Gamma$ discrete, then $\hat{G} = \text{Hom}(\Gamma, \mathbb{T})$ is compact and a subgroup $A$ separates the points of $G$ if and only if $A \subset \hat{G}$ is dense.
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• There exist a one-one relation between the above subgroups and the totally bounded topologies on $\Gamma$.

$$A_\zeta \mapsto \tau_\zeta.$$
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• There exist a one-one relation between the above subgroups and the totally bounded topologies on \(\Gamma\).

\[ A_\zeta \leftrightarrow \tau_\zeta. \]

• The relation above is order preserving:

\[ A \subset B \iff \tau_A \subset \tau_B. \]
Special case: $\Gamma = \mathbb{Z}$

- If $G = \mathbb{Z}$ (discrete), then $\hat{G} = \text{Hom}(\Gamma, \mathbb{T}) = \mathbb{T}$. This group has $2^\mathfrak{c}$-many dense subgroups. Hence $\mathbb{Z}$ has $2^\mathfrak{c}$-many totally bounded topologies.
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- If $G = \mathbb{Z}$ (discrete), then $\hat{G} = \text{Hom}(\Gamma, \mathbb{T}) = \mathbb{T}$. This group has $2^\omega$-many dense subgroups. Hence $\mathbb{Z}$ has $2^\omega$-many totally bounded topologies.

- If $A$ is a dense subgroup of $\mathbb{T}$, then $\tau_A$ denotes the weakest topology on $\mathbb{Z}$ such that the maps

  $$n \mapsto na \mod 1$$

  are continuous, for each $a \in A$. 
Diagram, \( N = A(b\mathbb{Z}, A) \)

\[\mathbb{Z}_A := (\mathbb{Z}, \tau_A)\]

\[
\begin{array}{cccccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z}^+ & \longrightarrow & \mathbb{Z}_A & = \mathbb{Z}_A \\
\downarrow & & \downarrow & & \downarrow & \\
 b\mathbb{Z} & \longrightarrow & b\mathbb{Z}/N & = \overline{\mathbb{Z}_A} \\
\end{array}
\]

\[
\begin{array}{cccccc}
\mathbb{T} & \longleftarrow & \mathbb{T}_d & \longleftarrow & \overline{b\mathbb{Z}/N} & = A \\
\end{array}
\]
Raczkowski (1998)

• There exist $2^c$-many dense non measurable subgroups $A_\zeta$ of $\mathbb{T}$ of cardinality $c$, hence
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- there exist $2^c$-many totally bounded topologies $\tau_\zeta$ of weight $c$ on $\mathbb{Z}$ under which the only compact subsets of $\mathbb{Z}_{A_\zeta}$ are the finite ones.
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• there exist $2^c$-many totally bounded topologies $\tau_\zeta$ of weight $c$ on $\mathbb{Z}$ under which the only compact subsets of $\mathbb{Z}_{A_\zeta}$ are the finite ones.

• There also exist $2^c$-many totally bounded topologies $\tau_\zeta$ of weight $c$ on $\mathbb{Z}$ under which the sequence $n! \longrightarrow 0$. 
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Notice that if $A_\zeta := (\mathbb{Z}, \tau_\zeta)$, then $mA_\zeta = 0$. 
Raczkowski’s question

• **Question.** Is there a subgroup $A$ of $T$ of measure $0$, such that the only compact subsets of $Z_A$ are finite?
Answers to Raczkowski’s question

• **Question.** Is there a subgroup $A$ of $\mathbb{T}$ of measure 0, such that the only compact subsets of $\mathbb{Z}_A$ are finite?

• (Barbieri, Dikranjan, Milan & Weber 2003). Yes under Martin’s axiom.
Answers to Raczkowski’s question

- **Question.** Is there a subgroup $A$ of $\mathbb{T}$ of measure 0, such that the only compact subsets of $\mathbb{Z}_A$ are finite?

- (Barbieri, Dikranjan, Milan & Weber 2003). Yes under Martin’s axiom.

- (Hart & Kunen 2005). Yes.
Answers to Raczkowski’s question

- **Question.** Is there a subgroup $A$ of $T$ of measure 0, such that the only compact subsets of $\mathbb{Z}_A$ are finite?

- (Barbieri, Dikranjan, Milan & Weber 2003). Yes under Martin’s axiom.

- (Hart & Kunen 2005). Yes.

- Both subgroups above have cardinality $\mathfrak{c}$. 
Question 1.

- Is there a measurable subgroup $A$ of $T$ of cardinality less than $c$, such that the only compact subsets of $\mathbb{Z}_A$ are finite? In particular...
Question 1 ($\aleph_1$)

- Is there a measurable subgroup $A$ of $\mathbb{T}$ of cardinality $\aleph_1$, such that the only compact subsets of $\mathbb{Z}_A$ are finite?
Question 1 ($\aleph_1$)

- Is there a measurable subgroup $A$ of $T$ of cardinality $\aleph_1$, such that the only compact subsets of $\mathbb{Z}_A$ are finite?

- Remarks.
  - We are assuming $\neg$CH
Question 1 ($\aleph_1$)

- Is there a measurable subgroup $A$ of $\mathbb{T}$ of cardinality $\aleph_1$, such that the only compact subsets of $\mathbb{Z}_A$ are finite?

- Remarks.
  - We are assuming $\neg\text{CH}$
  - There are models of $\neg\text{CH}$ in which there exist nonmeasurable subgroups of $\mathbb{T}$ of cardinality $\aleph_1$. 

[p. 59/91]
Question 1 (ℵ₁)

- Is there a measurable subgroup \( A \) of \( T \) of cardinality \( ℵ₁ \), such that the only compact subsets of \( \mathbb{Z}_A \) are finite?

- Remarks.
  - We are assuming \( \neg \text{CH} \)
  - There are models of \( \neg \text{CH} \) in which there exist nonmeasurable subgroups of \( T \) of cardinality \( ℵ₁ \).
  - If the subgroup \( A \) of \( T \) is countable, then \( \mathbb{Z}_A \) is (totally bounded and) metrizable. Hence it has infinite compact sets.
Determined Subgroups

- If $H$ is a dense subgroup of $G$, then every $f \in \hat{H}$ extends to some $\overline{f} \in \hat{G}$.

\[
\begin{align*}
H & \subseteq \mathrel{\overrightarrow{\subseteq}} G \\
f & \downarrow \\
T & \mathrel{\overrightarrow{\equiv}} T
\end{align*}
\]
Determined Subgroups

• If $H$ is a dense subgroup of $G$, then every $f \in \hat{H}$ extends to some $\overline{f} \in \hat{G}$.

$$
\begin{array}{ccc}
H & \subseteq & G \\
\downarrow f & & \downarrow \overline{f} \\
\mathbb{T} & = & \mathbb{T}
\end{array}
$$
Also, if $f \in \hat{G}$ then its restriction to $H$ belongs in $\hat{H}$.

\[
\begin{array}{cccc}
H & \subset & G \\
\downarrow f_{\mid H} & & \downarrow f \\
\top & & \top
\end{array}
\]
Determined Subgroups

- Hence the map $\phi : f \mapsto f|_H$ is an isomorphism from $\hat{G}$ onto $\hat{H}$. 
Determined Subgroups

- Hence the map \( \phi : f \mapsto f|_H \) is an isomorphism from \( \hat{G} \) onto \( \hat{H} \).
- Since compact subsets of \( H \) are compact in \( G \)

\[
\phi : \hat{G} \longrightarrow \hat{H}
\]

is continuous in the compact-open topology.
Determined Subgroups

- Hence the map $\phi : f \mapsto f|_H$ is an isomorphism from $\hat{G}$ onto $\hat{H}$.
- Since compact subsets of $H$ are compact in $G$

$$\phi : \hat{G} \longrightarrow \hat{H}$$

is continuous in the compact-open topology.

- **Question.** When is $\phi : \hat{G} \longrightarrow \hat{H}$ a topological isomorphism?
Determined Subgroups

- **Definition (CRT, 2001).** If $H$ is a dense subgroup of $G$, we say that $H$ determines $G$ if $\phi : \hat{G} \longrightarrow \hat{H}$ is a topological isomorphism.
Determined Subgroups

- **Definition** (CRT, 2001). If $H$ is a dense subgroup of $G$, we say that $H$ determines $G$ if $\phi : \hat{G} \rightarrow \hat{H}$ is a topological isomorphism.

- We say that $G$ is determined if every dense subgroup of $G$ determines $G$. 
Determined Subgroups

• **Definition (CRT, 2001).** If $H$ is a dense subgroup of $G$, we say that $H$ determines $G$ if $\phi : \hat{G} \to \hat{H}$ is a topological isomorphism.

• We say that $G$ is **determined** if every dense subgroup of $G$ determines $G$.

• $\mathbb{Z}^+$ does not determine $b\mathbb{Z}$. 
Determined Subgroups

- **Definition (CRT, 2001).** If $H$ is a dense subgroup of $G$, we say that $H$ determines $G$ if $\phi : \hat{G} \longrightarrow \hat{H}$ is a topological isomorphism.

- We say that $G$ is determined if every dense subgroup of $G$ determines $G$.

- $\mathbb{Z}^+$ does not determine $b\mathbb{Z}$.
  For $\hat{\mathbb{Z}}^+ = \mathbb{T}$, whereas $\hat{b\mathbb{Z}} = \mathbb{T}_d$. 
Few results

- (Chasco-Aussenhofer, 1998) Every metrizable group is determined.
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• (Chasco-Aussenhofer, 1998) Every metrizable group is determined.

• (CRT, 2001) A compact group of weight $\mathfrak{c}$ (or bigger) is not determined.
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  - For example, $\mathbb{T}^c$ is not determined.
Few results

• (Chasco-Aussenhofer, 1998) Every metrizable group is determined.

• (CRT, 2001) A compact group of weight $\mathfrak{c}$ (or bigger) is not determined.
  • For example, $\mathbb{T}^\mathfrak{c}$ is not determined.
  • $\mathbb{Z}_n^\mathfrak{c}$ is not determined.
Question 2. (CRT, 2001)

• Is a compact group of weight $< c$ determined? In particular...
Question 2 ($\aleph_1$). (CRT, 2001)

- Is a compact group of weight $\aleph_1$ determined?
Question 2 ($\aleph_1$). (CRT, 2001)

• Is a compact group of weight $\aleph_1$ determined?
• We are assuming $\neg$CH
Interplay between questions 1 and 2.

- Let $A$ be a measurable subgroup of $T$ of cardinality less than $\aleph$, such that the only compact subsets of $\mathbb{Z}_A$ are finite.
Interplay between questions 1 and 2.

- Let $A$ be a measurable subgroup of $\mathbb{T}$ of cardinality less than $\aleph$, such that the only compact subsets of $\mathbb{Z}_A$ are finite.

- Set $K = \overline{\mathbb{Z}_A}$. Then
Interplay between questions 1 and 2.

• Let $A$ be a measurable subgroup of $T$ of cardinality less than $\mathfrak{c}$, such that the only compact subsets of $\mathbb{Z}_A$ are finite.

• Set $K = \overline{\mathbb{Z}_A}$. Then

• $K$ is a compact group of weight $|A| < \mathfrak{c}$,
Interplay between questions 1 and 2.

- Let $A$ be a measurable subgroup of $T$ of cardinality less than $c$, such that the only compact subsets of $\mathbb{Z}_A$ are finite.
- Set $K = \overline{\mathbb{Z}_A}$. Then
- $K$ is a compact group of weight $|A| < c$,
- $\mathbb{Z}_A$ is a dense subgroup of $K$, 
Interplay between questions 1 and 2.

• Let \( A \) be a measurable subgroup of \( \mathbb{T} \) of cardinality less than \( c \), such that the only compact subsets of \( \mathbb{Z}_A \) are finite.

• Set \( K = \overline{\mathbb{Z}_A} \). Then

• \( K \) is a compact group of weight \( |A| < c \),

• \( \mathbb{Z}_A \) is a dense subgroup of \( K \),

• \( \hat{\mathbb{Z}_A} = A \subset \mathbb{T} \), whereas
Interplay between questions 1 and 2.

- Let $A$ be a measurable subgroup of $\mathbb{T}$ of cardinality less than $\mathfrak{c}$, such that the only compact subsets of $\mathbb{Z}_A$ are finite.

- Set $K = \overline{\mathbb{Z}_A}$. Then

- $K$ is a compact group of weight $|A| < \mathfrak{c}$,

- $\mathbb{Z}_A$ is a dense subgroup of $K$,

- $\hat{\mathbb{Z}_A} = A \subset \mathbb{T}$, whereas

- $\hat{K} = A_d$. 
Interplay between questions 1 and 2.

- Let \( A \) be a measurable subgroup of \( \mathbb{T} \) of cardinality less than \( c \), such that the only compact subsets of \( \mathbb{Z}_A \) are finite.
- Set \( K = \overline{\mathbb{Z}_A} \). Then
  - \( K \) is a compact group of weight \( |A| < c \),
  - \( \mathbb{Z}_A \) is a dense subgroup of \( K \),
  - \( \hat{\mathbb{Z}_A} = \hat{A} \subset \mathbb{T} \), whereas
  - \( \hat{K} = \hat{A}_d \).
- Therefore \( K \) is a compact group of weight \( < c \) that is not determined.
Interplay between questions 1 and 2 \((\aleph_1)\).

- If \(A\) is a measurable subgroup of \(\mathbb{T}\) of cardinality \(\aleph_1\), such that the only compact subsets of \(\mathbb{Z}_A\) are finite, then there is a compact group of weight \(\aleph_1\) that is not determined.
Few more questions.

• Is $T_{\aleph_1}$ non-determined?
Few more questions.

- Is $\mathbb{T}^\aleph_1$ non-determined?
- If $F$ is a finite group, is $F^\aleph_1$ non-determined?
Few more questions.

- Is $\mathbb{T}^{\aleph_1}$ non-determined?
- If $F$ is a finite group, is $F^{\aleph_1}$ non-determined?

Last two questions are equivalent.
Few more questions.

- Is $\mathbb{T}^{\aleph_1}$ non-determined?
- If $F$ is a finite group, is $F^{\aleph_1}$ non-determined? Last two questions are equivalent.
- If $G_1$ and $G_2$ are determined topological groups, is $G_1 \times G_2$ determined?
Few more questions.

• Is $\mathbb{T}^{\aleph_1}$ non-determined?
• If $F$ is a finite group, is $F^{\aleph_1}$ non-determined?
  Last two questions are equivalent.
• If $G_1$ and $G_2$ are determined topological groups, is $G_1 \times G_2$ determined? In particular:
  • If $G$ is determined, is $G \times G$ determined?
Respectfully dedicated

To Ta-Sun Wu

In memoriam

“Those fortunate enough to have known him personally will miss his smile and his friendly, self-effacing humility. The profession has been deprived of a significant figure of great creativity.”

WWC and SH