

A CHARACTERIZATION OF PONTRYAGIN-VAN KAMPEN DUALITY FOR COMPLEX LOCALLY CONVEX SPACES

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ABSTRACT. Topological vector spaces (TVS) are topological Abelian groups when considered under the operation of addition. It is therefore natural to ask when they satisfy Pontryagin-van Kampen (P-vK) duality. A number of attempts have been considered for *real* LCS. In particular, in a previous paper we showed a characterization of (P-vK) duality for real LCS. In this article we show that this characterization also holds for complex spaces.

1. INTRODUCTION AND MOTIVATION

If G is an Abelian topological group, a *character* of G is a continuous group homomorphism from G into the unit circle \mathbb{T} , the latter equipped with the topology inherited from the usual complex plane. Thus \widehat{G} , the *character group* of G , is defined by

$$\widehat{G} := \{h : G \rightarrow \mathbb{T} \mid h \text{ is a character}\},$$

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with group operation defined pointwise:

$$(h_1 h_2)(x) := h_1(x) h_2(x) \quad \forall x \in G.$$

If we put on \widehat{G} the *compact-open topology* [c-o], *i.e.*, the topology whose basic open sets are of the form

$$(K, O) := \{h \in \widehat{G} : h[K] \subset O\},$$

where $K \subset G$ is compact, and $O \subset \mathbb{T}$ is open, then $X := \widehat{G}[\text{c-o}]$ turns out to be a topological Abelian group.

We say that G satisfies *group duality* if the map

$$\Omega : G \longrightarrow \widehat{X}[\text{c-o}]$$

defined by

$$\Omega(g)(h) := h(g) \quad \forall g \in G,$$

is a topological isomorphism. The celebrated theorem of *Pontryagin-van Kampen* states that every locally compact Abelian group satisfies group duality. When G satisfies group duality, we denote this situation by writing $G \in \mathcal{D}$. The class just defined contains also some non locally compact groups since it is closed under arbitrary products, as proved by KAPLAN [9].

A topological vector space (TVS) equipped with the sum as operation is *a priori* a topological Abelian group. As such, it is natural to ask if it satisfies group duality. The first attempt in doing so was done by SMITH [13] who proved that Banach spaces and reflexive locally convex spaces (LCS) over \mathbb{R} satisfy group duality (see the comment after Theorem 3.3 below). Later WATERHOUSE [14] extended this result to quasi-complete barrelled spaces (in

particular Fréchet and reflexive spaces) over a field having a non-trivial absolute value.

For the case that concerns us in this paper, namely for complex LCS, some other authors (*e.g.*, [1], [4], [5], & [10] (§23.9)) have independently made contributions to the SMITH'S program, sometimes considering a different topology on the character groups, some others giving only sufficient conditions for a LCS to satisfy group duality. The problem of finding also necessary conditions was overtaken recently by KYE [11] who offered a full characterization of those *real* LCS that satisfy group duality. However some of his proofs are mistaken and it is an open question whether or not his main result stands. In an effort to correct it, the present authors have offered in [6] an alternative characterization of those *real* LCS that satisfy group duality. The goal in this short note is to show that the main results in [6] also hold for complex LCS.

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2. NOTATION AND PRELIMINARIES

The symbols \mathbb{R} , \mathbb{C} , and \mathbb{T} will stand for real numbers, complex numbers, and the unit circle, respectively, each equipped with its usual topology. The unit disk \mathbb{D} is defined as $\{z \in \mathbb{C} : |z| \leq 1\}$. E denotes a linear space over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$, and t denotes a Hausdorff topology on E such that $E[t]$ is a topological vector space. We specialize to the case when $E[t]$ is locally convex. E' denotes the vector space (over \mathbb{K}) of all continuous linear

functions from $E[t]$ to \mathbb{K} , *i.e.*, the functionals of $E[t]$; E' is also known as the *topological dual* of $E[t]$. If $A \subseteq E$, the *polar of A in E'* , denoted by A° , is defined by

$$A^\circ := \{f \in E' : |f(a)| \leq 1 \ \forall a \in A\}.$$

If $A \subseteq E'$, then the *polar A° of A in E* , is defined in an obvious similar way. The symbol σ denotes the weak topologies on E and E' ; that is: On E σ denotes the weakest topology that makes the elements of E' continuous, and on E' it denotes the topology of pointwise convergence (also known as the *weak- $*$ topology*), also characterized as the (necessarily locally convex) topology whose zero neighborhoods are the polars of the finite subsets of E . $A \subset E[t]$ is said to be *weakly compact* if it is a compact subset of $E[\sigma]$. The symbol τ_c denotes the (locally convex) topology on E' with zero neighborhoods given by the polars of the t -compact subsets of E . More generally, if $\langle E, F \rangle$ denotes a *dual pair* [10] (§10.3), and \mathfrak{M} denotes a collection of subsets of F , then $t_{\mathfrak{M}}$ denotes the *topology on E of uniform convergence on \mathfrak{M}* (see [10], §21). For example, if \mathfrak{M} denotes the finite (t -compact, resp.) subsets of E , then $t_{\mathfrak{M}} = \sigma$ ($t_{\mathfrak{M}} = \tau_c$, resp.).

Of fundamental importance in what follows is §21.11 of KÖTHE [10]: In particular, if $\mathbb{K} = \mathbb{C}$ and if $E[t]$ is given as above, we denote it by $E_r[t]$ when considered as a TVS over the real numbers. If $u \in (E_r)'$, then the function $c(u) : E \rightarrow \mathbb{C}$ defined by $c(u)(x) := u(x) - iu(ix)$ is complex-linear and t -continuous, *i.e.*, $c(u) \in E'$, and moreover, every $v \in E'$ has the form $c(u)$ for suitable $u \in (E_r)'$.

A straightforward calculation shows that

Proposition 2.1. *Let \mathfrak{M} denote a collection of subsets of E such that*

$M \in \mathfrak{M} \implies iM := \{im : m \in M\} \in \mathfrak{M}$. Then the map

$c : (E_r)'[t_{\mathfrak{M}}] \longrightarrow E'[t_{\mathfrak{M}}]$ is a topological group isomorphism.

In particular, if $\theta \in \{\sigma, \tau_c\}$, then the map $c : (E_r)'[\theta] \longrightarrow E'[\theta]$ is a topological group isomorphism. ■

We must be careful in explicitly pointing out whether we are taking real or complex polars, as often we will be viewing the space both ways.

Notice that if $u \in (E_r)'$, then $h_u : E[t] \longrightarrow \mathbb{T}$, defined by

$$h_u(x) := e^{2\pi i u(x)} \quad \forall x \in E,$$

defines an element in \widehat{E} . That the map $u \mapsto h_u$ from $(E_r)'$ into \widehat{E} is an algebraic group isomorphism follows from a result independently proved by HEWITT and ZUCKERMAN [7] (Theorem 6) and SMITH [13] (Lemma 1). By using WATERHOUSE [14] (Theorem 2) instead we can further state

Proposition 2.2. *The map $u \mapsto h_u$ from $(E_r)'$ into \widehat{E} is a topological group isomorphism from $(E_r)'[\tau_c]$ onto $\widehat{E}[c\text{-o}]$. ■*

Denoting by $\Re z$ the real part of $z \in \mathbb{C}$, we obtain:

Corollary 2.3. *The map $v \mapsto h_{\Re v}$ from E' into \widehat{E} is a topological group isomorphism from $E'[\tau_c]$ onto $\widehat{E}[c\text{-o}]$. ■*

Properties C1 and Ascoli. We will say that the LCS $E[t]$ satisfies

Property C1 if the balanced, convex and closed hull (bcch) of any t -compact subset of E is weakly compact.

As usual, if $A \subseteq E'$, then we say that A is t -*equicontinuous* if for every $\varepsilon > 0$, the set $\bigcap_{v \in A} v^{-1}[B_\varepsilon(0)]$ is a t -neighborhood of 0 in E , where $B_\varepsilon(0)$ denotes the open ball around 0 in \mathbb{K} of radius ε and $v^{-1}[B_\varepsilon(0)]$ its preimage.

We will say that $E[t]$ satisfies *Ascoli* if any τ_c -compact subset of E' is t -equicontinuous.

The characterization of a *real* LCS that satisfies group duality is given in [6] by:

Theorem 2.4. *Assume $E[t]$ is a LCS over \mathbb{R} . Then $E[t] \in \mathcal{D}$ if and only if it satisfies C1 and Ascoli.* ■

3. CHARACTERIZATION OF COMPLEX LCS $\subset \mathcal{D}$

In this section $E[t]$ denotes a complex LCS. If $S \subseteq E$, we denote by $\text{bcch}_r(S)$ the balanced, convex and closed hull of S as a subspace of $E_r[t]$.

Lemma 3.1. *$E[t]$ satisfies C1 if and only if $E_r[t]$ satisfies C1.*

Proof: Assume $E[t]$ satisfies C1, and let K be a compact subspace of $E_r[t]$. Since clearly $\text{bcch}_r(K) \subseteq \text{bcch}(K)$ and the latter is weakly compact, the result follows as the topological groups $E[\sigma]$ and $E_r[\sigma]$ are topologically isomorphic [10] (§21.11(2)).

Assume $E_r[t]$ satisfies C1, and let K be a compact subspace of $E[t]$. Plainly $K \subseteq \text{bcch}_r(K) \subseteq \mathbb{D} \cdot \text{bcch}_r(K) \subseteq \text{bcch}_r(K) + i\text{bcch}_r(K)$, and the latter is easily seen to be closed, convex, and balanced in $E[t]$. Hence $\text{bcch}(K) \subseteq \text{bcch}_r(K) + i\text{bcch}_r(K)$. Since $\text{bcch}_r(K)$ is assumed to be weakly compact, the result follows again from the fact that $E[\sigma]$ and $E_r[\sigma]$ are topologically isomorphic groups. ■

Lemma 3.2. *$E[t]$ satisfies Ascoli if and only if $E_r[t]$ satisfies Ascoli.*

Proof: Assume $E[t]$ satisfies Ascoli; let A be a compact subspace of $(E_r)'\tau_c$, and choose $\varepsilon > 0$. By Proposition 2.1 $c[A] = \{c(u) \in E' : u \in A\}$ is a compact subspace of $E'\tau_c$. Since $\bigcap_{v \in c[A]} v^{-1}[B_\varepsilon(0)] \subseteq \bigcap_{u \in A} u^{-1}[B_\varepsilon(0)]$ and the former is a t -neighborhood of 0 in E , it follows that $\bigcap_{u \in A} u^{-1}[B_\varepsilon(0)]$ is a t -neighborhood of 0 in E_r as well.

Assume $E_r[t]$ satisfies Ascoli; let A be a compact subspace of $E'\tau_c$, and choose $\varepsilon > 0$. By Proposition 2.1 $A_r := c^{-1}[A] = \{u \in E_r' : c(u) \in A\}$ is a compact subspace of $(E_r)'\tau_c$. Since $\bigcap_{u \in A_r} u^{-1}[B_{\varepsilon/\sqrt{2}}(0)] \subseteq \bigcap_{v \in A} v^{-1}[B_\varepsilon(0)]$ and the former is a t -neighborhood of 0 in E_r , it follows that the latter is also a t -neighborhood of 0 in E , as required. ■

Theorem 3.3. *Let $E[t]$ be a LCS over the complex field. The following assertions are pairwise equivalent.*

- (a) $E[t] \in \mathcal{D}$,
- (b) $E[t]$ satisfies C1 and Ascoli,
- (c) $E_r[t] \in \mathcal{D}$, and
- (d) $E_r[t]$ satisfies C1 and Ascoli.

Proof: (a) and (c) are equivalent since $E[t]$ and $E_r[t]$ are topologically isomorphic groups. (c) \iff (d) is Theorem 2.4. (b) \iff (d) follows from Lemmas 3.1 and 3.2. ■

As noticed already, for Banach or reflexive spaces over \mathbb{R} the following result is due to SMITH [13], although her proofs work also for the complex case by replacing the map $u \mapsto h_u$ from E' to \widehat{E} in the real case by the map $u \mapsto h_{\mathcal{R}u}$

of Corollary 2.3. As it stands below, it was proved by WATERHOUSE [14] (Corollary to Theorem 3). Another proof, for real spaces, was given in [6] (Corollary 2).

Corollary 3.4. *Fréchet and reflexive spaces satisfy group duality.*

Proof: If $E[t]$ is as hypothesized, then so is $E_r[t]$. Now apply Corollary 2 in [6] with Theorem 3.3. ■

4. PROPERTY C2

Recall that in a locally convex space any absorbent, absolutely convex and closed subset is said to be a *barrel* (see [10] (p. 257)). The concept that is different in the real case *vis-a-vis* the complex case is that of absolute convexity. If $E[t]$ happens to be a complex LCS, then the barrels of $E_r[t]$ are referred to as the *r-barrels* of $E[t]$. Clearly any barrel is a r-barrel. Lemma 4.3 *infra*, shows that any r-barrel contains a barrel.

Definition 4.1. We say that $W \subset E[t]$ is a k_σ -neighborhood of 0 if for any t -compact subset $K \subset E$ containing 0, there exists a σ -neighborhood U of 0 such that $U \cap K \subset W \cap K$. The symbol t_{k_σ} will denote the topology on E whose zero neighborhoods are the barrels which are also k_σ -neighborhoods of zero.

In other words t_{k_σ} is the maximal locally convex *polar* topology such that $E[t_{k_\sigma}]$ and $E[t]$ have the same compact subsets. For, $t \subseteq t_{k_\sigma}$, and t_{k_σ} lies between t and its k -group extension $k_g(t)$ [12] (Prop. 1).

Next we prove a technical result to be used in the sequel.

Lemma 4.2. *Let $E[t]$ be a complex LCS, and assume that B is a r -barrel.*

Then

(a) *iB is a r -barrel,*

(b) *$B \cap iB$ is a r -barrel,*

(c) *if B is a k_σ -neighborhood of zero, then $B \cap iB$ is also a k_σ -neighborhood of zero.*

Proof: (a) and (b) follow by observing that the map $i : E[t] \longrightarrow E[t]$, defined by $i(x) := ix$, is a topological isomorphism of real vector spaces.

(c) If K is a t -compact subset of E containing 0, then so is $-iK$, hence there are σ -neighborhoods U and V of 0 such that $U \cap -iK \subset B \cap -iK$ and $V \cap K \subset B \cap K$. By [10] (§21.11(2)) we can choose U to be complex-balanced. Then a quick check proves $(U \cap V) \cap K \subset K \cap (B \cap iB)$ as required. ■

Lemma 4.3. *If $E[t]$ is a complex LCS and B is a r -barrel such that $B = iB$, then $\text{bcch}(\frac{1}{2}B)$ is a barrel contained in B . In particular, any r -barrel contains a barrel.*

Proof: Clearly $\text{bcch}(\frac{1}{2}B)$ is a barrel. The first part follows since $\text{bcch}(\frac{1}{2}B)$ is the closure of the convex cover of $\cup_{|\lambda| \leq 1} \lambda(\frac{1}{2}B)$ (see [10] §16.1(2)), and the latter is easily seen to be a subset of B . The remaining part follows from (b) of the lemma above. ■

The following result resembles [10] (§21.11.(2)).

Corollary 4.4. *If $E[t]$ is a complex LCS, then the identity map $I : E[t_{k_\sigma}] \longrightarrow E_r[t_{k_\sigma}]$ is a topological isomorphism.*

Proof: Since barrels are r -barrels, I is clearly open. To see continuity, let B be a barrel in $E_r[t]$ (a r -barrel) that is a k_σ -neighborhood of 0. By Lemma 4.2 we can take $B = iB$, and by the lemma above, $\text{bcch}(\frac{1}{2}B)$ is a barrel contained in B .

To finish the proof notice that $\text{bcch}(\frac{1}{2}B)$ is itself a k_σ -neighborhood of 0 as it is easy to see that multiples of k_σ -neighborhoods of 0 are k_σ -neighborhoods of 0 as well. ■

Definition 4.5. We say that $E[t]$ satisfies *Property C2* if any barrel that is a k_σ -neighborhood of zero is also a t -neighborhood of zero; that is, if $t = t_{k_\sigma}$.

By Lemma 7 in [6], real LCS with C2 satisfy Ascoli, hence from Lemma 3.2 it follows that *in complex LCS C2* \implies *Ascoli*.

The following is the most important result of this section:

Theorem 4.6. *If $E[t]$ satisfies C1 and C2, then $E[t] \in \mathcal{D}$.*

Proof: As mentioned above C2 \implies Ascoli. Now apply Theorem 3.3. ■

As usual, a TVS is said to be *quasi-complete* if every bounded closed subset is complete. It follows from Proposition 2.1 that:

Lemma 4.7. *$E'[\tau_c]$ is quasi-complete if and only if $(E_r)'[\tau_c]$ is quasi-complete.* ■

We have the following partial converse of Theorem 4.6:

Theorem 4.8. *If $E[t] \in \mathcal{D}$ and $E'[\tau_c]$ is quasi-complete, then $E[t]$ satisfies C1 and C2.*

Proof: By Theorem 3.3 $E_r[t]$ satisfies C1, and Lemma 4.7 assures that $(E_r)'\tau_c$ is quasi-complete. Hence Theorem 3 in [6] implies that $E_r[t]$ satisfies C1 and C2. Then Lemma 3.1 shows that $E[t]$ satisfies C1 and from the definition of Property C2 it is easy to see that if $E_r[t]$ satisfies C1 then so does $E[t]$. ■

Question 4.9. *Can the quasi-complete requirement in Theorem 4.8 be dropped?*

5. CLASSES OF LCS SATISFYING GROUP DUALITY

The goal of this last section is to show that, as in the real case, there is a wide range of complex locally convex spaces that satisfy group duality. We do this as Theorem 5.5. Recall that a LCS $E[t]$ is said to be *barrelled* if every barrel is a t -neighborhood of 0. A subset B of $E[t]$ is called *bornivorous* if B absorbs every bounded subset. Then $E[t]$ is said to be *bornological* if every bornivorous barrel is a t -neighborhood of 0.

Proposition 5.1. (1) *If $E[t]$ is bornological, then E' is τ_c -complete.*
 (2) *If $E[t]$ is barrelled, then E' is weakly quasi-complete.*

Proof: Statement 1 follows from [3] (IV, p. 5, rem. 2) and [10] (p. 385); statement 2 is [10] (§23.1(3)). ■

Lemma 5.2. *If $E[t]$ is a barrelled space and satisfies Property C1, then $E'\tau_c$ is quasi-complete.*

Proof: By [10] (§21.11.(2)) $E_r[t]$ is barrelled, and by Lemma 3.1 $E_r[t]$ satisfies Property C1. Hence Lemma 8 in [6] implies that $(E_r)'[\tau_c]$ is quasi-complete. Now apply Lemma 4.7. ■

Lemma 5.3. *If $E[t]$ is a complex LCS and B is a bornivorous r -barrel, then it contains a bornivorous barrel.*

Proof: By Lemma 4.2, we can assume that $B = iB$. That $\text{bcch}(\frac{1}{2}B)$ is a barrel follows from Lemma 4.3, which also assures that $\frac{1}{2}B \subseteq \text{bcch}(\frac{1}{2}B) \subseteq B$. Being B bornivorous implies the same for $\frac{1}{2}B$, hence $\text{bcch}(\frac{1}{2}B)$ is as required. ■

The following result is quite similar to [10] (§21.11.(2)) and Lemma 4.4. We skip its easy proof.

Lemma 5.4. *If $E[t]$ is a complex LCS, then $E[t]$ is bornological if and only if $E_r[t]$ is also bornological.* ■

The following result generalizes [5] (Theorem 1), [14] (Theorem 3), and [6] (Theorem 4) from which it follows:

Theorem 5.5. *If $E[t]$ is barrelled or bornological, and satisfies Property C1, then $E[t] \in \mathcal{D}$.*

Proof: By [6] (Theorem 4) the result is true if the space is real. Otherwise, by Theorem 4.6 we are only left to prove that $E[t]$ satisfies C2. By [10] (§21.11.(2)) and the last lemma, respectively, $E_r[t]$ is barrelled or bornological, hence by [6] (Theorem 4) $E_r[t] \in \mathcal{D}$. Now apply Theorem 3.3. ■

In particular, this theorem implies, as a special case, that *quasi-complete barreled spaces satisfy duality*, a result first proved by WATERHOUSE [14]. That the converse of the preceding theorem does not hold, is witnessed by Examples 7 and 8 in [6]. To see that Property C1 cannot be dismissed either, see Example 9 in [6]. Although those examples are real, they can easily be re-defined as complex spaces.

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