The Wigner crystal is a charged particle system in two dimensions (2D) which is practically attracting a great deal of research interest (see, for example some conference proceedings 1,2). In the case of electrons or helium ions on the surface of liquid helium, where the typical surface concentrations are $n < 10^{-11}$ cm$^{-2}$, 2D Wigner crystal of electrons or helium ions is realized under classical conditions. In the case of 2D semiconductor systems, in which the typical electron concentrations are $n > 10^{-11}$ cm$^{-2}$, the possibility of using a high magnetic field to induce crystallization under ultracold conditions has been realized for high-mobility electrons in GaAs/GaAlAs heterostructures. This possibility had been discussed theoretically many years ago. Evidence that this possibility has been realized has come from experiments by G. spectroscopy, a magnetotransport method (see also Refs. 1 and 2). In these systems, under quantum conditions, with a partial filling of the lowest Landau level ($\nu = nh/eF < 1$), by electrons, a competition occurs between the state with a crystalline order and the state of an incompressible quantum fluid of the Laughlin type near the fractional values $\nu = 1/5$, 1/7, 1/9. As the magnetic field $B$ is raised further, quantum fluctuations are suppressed, and at fixed values of the electron concentration $n$ and the temperature $T$ a Wigner crystal should be realized under classical conditions. At sufficiently small values of $\nu$, a classical regime prevails near the melting line. This regime may have been realized in the experiments of Ref. 8 (with $\nu < 0.1$).

The actual experiments of Ref. 9 and the numerical simulations of Ref. 10 indicate that the melting of a classical 2D Wigner crystal occurs by a Kosterlitz-Thouless dislocation mechanism 11-14 and that it is a continuous phase transition, not accompanied by abrupt changes in thermodynamic properties. In the present paper we examine the effect of a frozen-in disorder on the translation order, on the orientation order, and on the dislocation melting of a 2D Wigner crystal. Some of the results below were reported in Ref. 15. The dislocation-melting model has been used previously to study certain problems concerning 2D crystals with a short-range interaction between particles. Specifically, these problems were the effect of finite short-range impurities introduced in the crystal, the effect of a random substrate roughness, 16-18 and the effect of a random pinning force stemming from irregularities of the substrate. 19,20 The case of a 2D Wigner crystal requires consideration of two physically related circumstances. First, the long-range Coulomb force makes a 2D Wigner crystal incompressible in the long-range limit. Standard elastic theory, ordinarily employed in studying problems of this type, is thus not directly applicable in this case. Second, for a 2D Wigner crystal it is physically justified and indeed important to examine random fields with long-range correlations, e.g., a field of ionized donors.
critical state of a 2D Wigner crystal.
A random field corresponding to substrate irregularities is examined in Subsec. 3.5. This field interacts with a transverse phonon mode. As a result, the 2D Wigner crystal, like an ordinary 2D solid, is in a disordered state. Strictly speaking, a topological phase transition is impossible. However, if the order is disrupted by the random field over a distance of macroscopic size, then the behavior of the system may exhibit some rounded features associated with a dissociation of disorder lines.20

2. MODEL

In the continuum approximation the energy of static, homogeneous deformation of a 2D Wigner crystal with a triangular lattice is described by the Hamiltonian 21

\[ H_{\text{w}} = \int \frac{1}{2} \left( \partial_{\vec{r}} \psi^*(\vec{r}) \right)^2 + \frac{1}{2} \mu^2 \psi^*(\vec{r}) \psi(\vec{r}) + \frac{1}{2} \gamma^2 \psi^*(\vec{r}) \psi(\vec{r}) \]

\[ \times \left( \frac{\partial_{\vec{r}} \psi^*(\vec{r})}{\partial_{\vec{r}} \psi(\vec{r})} + \frac{\partial_{\vec{r}} \psi(\vec{r})}{\partial_{\vec{r}} \psi^*(\vec{r})} \right), \]

where \( u \) is the strain tensor; \( \mu \) and \( \gamma \) are elastic constants (Lamé coefficients), which we treat as phenomenological constants and which may, in particular, embody a dependence on the magnetic field \( B \) as \( \mu = \mu_0 (B/B_0) \); \( \gamma \) is the surface concentration of electrons; and \( x \) is the dielectric constant. The Hamiltonian \( H_{\text{w}} \) in (1) differs from the standard Hamiltonian of elastic theory in the presence of the last term, which corresponds to an energy component due to long-range Coulomb interactions of the charge density, \( \rho(\vec{r}) = ne(\vec{r}) \). These interactions arise because of the nontopological strain. It is thus possible to deal correctly with the incompressibility of the 2D Wigner crystal in the long-wave limit (see, for example expression (15) below for the Fourier transform of the dynamical matrix \( D_{jk}(q) \)).

The presence of a frozen disorder is described by the part of the Hamiltonian

\[ H_{\text{dis}} = \int \phi(\vec{r}) \psi(\vec{r})^* \psi(\vec{r}) \]

where \( f \) is the local density of the random force. We introduce two types of disorder in a phenomenological way, as described below (c.f. Refs. 16–20).

In the first case, the force \( f \) is related to the potential field \( \Phi(\vec{r}) \) created by the randomly distributed impurities, with a concentration \( c(\vec{r}) \):

\[ \psi^* \psi = \int \psi^* \psi(\vec{r} - \vec{r}') \phi(\vec{r}') d\vec{r}'. \]

This representation may be thought of as a continuation from a discrete Hamiltonian of the electron–impurity interactions with an effective potential \( v(\vec{r}) \) (e.g., a potential modified by image forces): 22

\[ \sum_{\vec{r}} \int \psi^* \psi(\vec{r} + \vec{r} - \vec{r}') d\vec{r}'. \]

Only those terms which are linear in the displacements \( u \) are retained. Here \( \Delta \) gives the positions of the sites of the ideal 2D lattice, and \( c(\vec{r}) = 2 \delta(\delta(\vec{r} - \vec{r}')) \) is the impurity concentration. The impurities may be in the crystal itself or in a 2D space.

Since we are interested in the long-range asymptotic behavior, we assume, regarding the impurities, that \( c(\vec{r}) \) is a Gaussian random function. We wish to stress that by writing the force \( f \) as the gradient of a potential field, \( f = -\Phi(\vec{r}) \) (or as the divergence of the stress field \( \sigma_{ij} \) due to the impurities, \( f_{ij} = \partial c_{ij} / \partial \vec{r} \)), by integrating by parts in (2), and by ignoring the boundary contribution, we can write Hamiltonian (2) as

\[ H_{\text{dis}} = \int \left( \frac{1}{2} \left( \partial_{\vec{r}} \Phi(\vec{r}) \right)^2 + \frac{1}{2} \gamma^2 \psi^*(\vec{r}) \psi(\vec{r}) \right) d\vec{r}. \]

In other words, the Hamiltonians which we have found for the interaction with a disorder are translationally invariant: They are unchanged by a shift of the lattice as a whole, \( \psi(\vec{r}) \rightarrow \psi(\vec{r} - \vec{u}) + \psi(\vec{r}) \) (see also the discussion in Refs. 16 and 19).

We will discuss two cases below.

1) The first is that of randomly distributed, frozen-in impurities which have been introduced in the 2D Wigner crystal. These impurities may participate in long-wave phonon displacements, but (as in Ref. 16) they cannot trade places with neighboring lattice sites over the duration of the experimental observation. For the case of a 2D Wigner crystal we assume that there may be a variety of electron–imurity interaction potentials \( \Phi(\vec{r}) \). In particular, there may be long-range potentials, which would distinguish this case from that of ordinary crystals, in which the interaction forces are of short range in the continuum approximation.

We consider the influence of effective potentials from fairly wide classes: (a) integrable potentials, i.e.,

\[ \Phi = \int \phi(\vec{r}) \psi(\vec{r}) d\vec{r}, \]

(\( \phi(\vec{r}) \) is the lattice constant); and (b) long-range potentials, for which we have the Fourier transform \( \Phi(q) = \phi(q) / q^2 \) in the limit \( q \rightarrow 0 \). This case corresponds to the behavior \( \Phi(r) \sim r^{-\alpha}, \) with \( 0 < \alpha < 1, \) as \( r \rightarrow \infty. \)

For the Fourier transform of the random force \( f(q) \) we have a Gaussian correlation function \( f^2 \)

\[ \langle f(q) f(q') \rangle = 2 \delta(q - q') \sigma_n^2 \Phi(q)^2, \]

where \( \sigma_n^2 \) is the impurity concentration (in particles per square centimeter). \( \Phi(q) \) is the Fourier transform of the potential \( \Phi(r) \), and the brackets \( \langle \cdots \rangle \) mean an average over the disorder.

2) The second case is that of excited donors which are distributed in a layer of thickness \( d \) as a minimal distance \( d \) from the plane of the 2D Wigner crystal. Assuming \( c_0, \Phi \rightarrow \infty \), we have a Gaussian correlation function similar to (3) in this case (see also Ref. 22).

\[ \langle f^2(q) \rangle = \sigma_n^2 \Phi(q)^2 \delta(q - q_n) \Phi(q_n)^2 \]

A disordered of another type is the random force \( f \) associated with irregularities of the substrate. For this force we assume, as in Refs. 18–20, a Gaussian distribution corresponding to a white noise:

\[ \langle f(0) f(0') \rangle = \sigma_n^2 \delta(\vec{r} - \vec{r}'), \]

\[ \langle f(q) f(q') \rangle = 0. \]

We wish to stress that the correlation functions in (3), (4), and (6) have different tensor structures (in addition to

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the different dependences on the wave vector \( q \): For fields corresponding to isoptic impurities, correlation functions (3) and (4) are purely longitudinal. As a result, random fields of this type interact exclusively with the longitudinal mode of the crystal, which for a 2D Wigner crystal is a plasma mode. This plasma mode is harder than the transverse phonon mode and thus has no effect at all on the orientational order in the harmonic approximation.

2. LONG-RANGE TRANSLATIONAL AND ORIENTATIONAL ORDERS

3.1. Correlation functions

The presence of a long-range translational order and of an orientational order can be established by analyzing the behavior of the following correlation functions, respective-

\[ C_\theta (r) = \langle \exp \left( \frac{i}{\hbar} \theta (\mathbf{r} - \mathbf{r}_0) \right) \rangle, \]

\[ C_{\phi}(r) = \langle \exp \left( \frac{i}{\hbar} \phi (\mathbf{r} - \mathbf{r}_0) \right) \rangle, \]

where \( C_\theta (r) \) is an arbitrary wave vector; \( \theta = \theta_0, \phi, \) and \( \phi \) is the 2D antisymmetric tensor. Two averages are taken in succession in (7) and (8): one over the thermodynamic ensemble at the temperature \( T \) (we are assuming \( \kappa = 1 \)) with the Hamiltonian \( H = H_0 + H_1 \) (this average is denoted by the angular brackets) and then one over the random field (this average is denoted by the square brackets).

The calculation in (7) and (8) is conveniently carried out in the Fourier representation, through the use of the relation

\[ \langle \exp (-iH_1F) \rangle \sim \langle \exp (-iH_1F) \rangle, \]

to make the transition to a calculation of thermodynamic averages \( \langle \cdots \rangle \) over the ensemble with the quadratic Hamiltonian \( H_1 \) from (1). As a result, Hamiltonian \( H_1 \) in (2) is linear in the displacements \( u \) (as are the arguments of the exponential functions of correlation functions (7) and (8)) and in the random variables \( f \) and since (b) the term quadratic in \( f \) in the numerator and denominator in (9) cancel out, all these averages reduce to Gaussian averages of the linear forms in the arguments of the exponential functions. For the correlation functions we readily find

\[ C_\theta (r) = C_\theta (0) \exp \left( -\frac{r^2}{\lambda_\theta} \right), \]

\[ C_{\phi}(r) = C_{\phi}(0) \exp \left( -\frac{r^2}{\lambda_{\phi}} \right), \]

where \( C_\theta (0) \) is the correlation function for the pure system, and \( C_{\phi}(0) \) is a multiplicative increment which stems from the disorder. For a purely 2D Wigner crystal we find the known statistical behavior \( r > 2 \lambda_\theta \).

\[ C_\theta (r) = r^{-\frac{1}{2}} G_{\eta}(r), \]

\[ G_{\eta}(r) = \frac{T}{4\pi q_0}, \]

\[ C_{\phi}(r) = \exp \left[ -\frac{2q_0}{4\pi |\phi|} r^2 \right]. \]

This behavior corresponds to a quasi-long-range-power-law translational order and to a genuinely long-range orientational order. A distinctive feature of (10) is that the only dependence is on the absolute value of the shear modulus \( \eta \) of the crystal. In a 2D Wigner crystal, thermal fluctuations of only the transverse phonon mode participate in the disruption of the translational order. This situation is closely related to the incompressibility as \( q = 0 \) (the hardness of the longitudinal plasma mode).

In a pure 2D system, a change in the asymptotic behavior of the correlation functions occurs (in the absence of an anharmonicity) only if topological defects—dislocations and disclinations—are taken into account. With \( T = \mu_\text{0} m_\text{0} \\eta / 4\pi, \) for example, the 2D Wigner crystal undergoes a topological phase transition involving a discontinuity of dislocation pairs. In the process it loses its static shear hardness: It melts into a biaxial liquid-crystal phase, in which a quasi-long-range orientational order prevails:

\[ C_{\phi}(r) = \exp \left( -\frac{r^2}{\lambda_{\phi}} \right), \]

\[ C_{\phi}(r) = r^{-\frac{1}{2}} G_{\eta}(r), \]

Here \( \xi(\ell) \) is a finite correlation length, which diverges in a power-law fashion as \( T \to T_c \).

If there is a disorder, the calculation (with phonons alone taken into account) leads to a power-law decay of \( C_{\phi}(r) \), then a topological phase transition is still possible, but in the pure system. The effect of a disorder on the latter transition in the case of a 2D Wigner crystal is analyzed below in Secs. 4 and 5.

If the disorder leads instead to an exponential decay of the correlation functions (with a length scale \( \ell \)), which definitely places a limit on the correlation length \( \xi(\ell) \), the meaning of fluctuations of the random field disrupt the crystal-line translational order. At low temperatures the 2D Wigner crystal is in a disordered "glasslike" state, and a topological phase transition cannot occur in x, strictly speaking (but see Subsec. 5 below and also Ref. 20).

3.2. Disorder-invoked multiplicative increments

For the increments in the correlation functions we have

\[ C_{\theta}(r) = \exp \left( -\frac{r^2}{\lambda_\theta} \right), \]

\[ C_{\phi}(r) = \exp \left( -\frac{r^2}{\lambda_{\phi}} \right), \]

\[ C_{\phi}(r) = \exp \left( -\frac{r^2}{\lambda_{\phi}} \right), \]

\[ C_{\phi}(r) = \exp \left( -\frac{r^2}{\lambda_{\phi}} \right), \]

where the dynamic matrix of the 2D Wigner crystal is

\[ D_{\phi}(q) = \mu_\text{0} (\lambda_\phi + \lambda_\phi + \lambda_\phi + \lambda_\phi) \psi_{\phi}(q), \]

\[ X(1 - \cos(q)) = \langle \psi_{\phi}(q) \rangle, \]

\[ X(1 - \cos(q)) = \langle \psi_{\phi}(q) \rangle, \]

where the transverse and longitudinal projection operators are given by the following expressions, respectively:

\[ \psi_{\phi}(q) = \frac{x^2 + \gamma^2}{\gamma^2 + \gamma^2}, \]

\[ \psi_{\phi}(q) = \frac{x^2 + \gamma^2}{\gamma^2 + \gamma^2}, \]

\[ \psi_{\phi}(q) = \frac{x^2 + \gamma^2}{\gamma^2 + \gamma^2}, \]

We are interested in the long-range asymptotic behavior of correlation functions (13) and (14) by their long-wave limits (in particular, for the Fourier transforms of the interaction potential \( V(q) \) introduced above).

Below we report results on the long-range asymptotic behavior of correlation functions (13) and (14) found in the ordinary harmonic approximation (in which dislocations
As above, the constant $C_{ij}$ incorporates the correlations in the arrangement of impurities in a phenomenological way. For the correlation function in (19) there are two regions which differ in the behavior of the correlation function. At $r > 2a$, we have a powerless-law decay: 

$$C_{ij}(r) \propto r^{-1}.$$ 

In the harmonic approximation, fluctuations of the random field of the charged donors that do not alter the power-law translational order. Note, however, that if $n_{max}$ is the quantity $n_{ij}$ (in 19) is by orders of magnitude small (for the principal reciprocal-lattice vector $q = 16a/2kF$, for example, we have $n_{ij} \approx 6.6 \times 10^{16}$), provided that the fluctuations of the donor charge are not suppressed in some special way (so that we have $C_{ij}(1)$), and provided that the relation $n_{max}$ is not realized in this system, for example by photocexcitation.

3.5. Random force exerted by the substrate

Using (6), and cutting off (as in Refs. 19 and 20) the integral in the argument of the exponential function in (13), which diverges as its lower limit, at the value corresponding to the reciprocal of the size of the system, $L^{-1}$, we find the following expression for the correlation functions describing the translational and orientational orders, respectively:

$$C_{ij}(q) \sim r^{-1} \exp \left[ -\frac{\gamma r_{1} r_{2}}{2} \right],$$

$$\gamma_{ij} \approx \frac{2\pi n_{ij}}{2k^{2}}.$$  

(21)
dislocation melting. In this case it is important to determine the effect of a disorder on this behavior, especially at low temperatures. For this purpose we bring dislocations into the discussion, writing the displacement as the sum of a dislocation part \( \mathbf{u}^{*} \) and a phonon part \( \tilde{\mathbf{p}} \):

\[
\mathbf{u}(\mathbf{r}) = \mathbf{u}^{*}(\mathbf{r}) + \tilde{\mathbf{p}}(\mathbf{r}) = \sum_{n} b_n \mathbf{b}_n \mathbf{z}_n + \sum_{\mathbf{k}} q_{\mathbf{k}}(\mathbf{r}) \tilde{\mathbf{p}}(\mathbf{r}),
\]

where \( b_n \) are the dimensionless Burgers vectors of the dislocations. As usual, we assume that the dislocations are at equilibrium:

\[
\frac{\partial}{\partial \mathbf{u}} \mathbf{u}^{*}(\mathbf{r}) = 0,
\]

where the Hamiltonian is \( H = H_b + H_{\perp} \). The component \( H_{\perp} \), from (2), corresponds to impurities which do not disrupt the power-law translational order in the harmonic approximation. By virtue of (23) and the quadratic dependence of \( H \) on the displacements, we have \( H(\mathbf{u}^{*} + \tilde{\mathbf{p}}) = H(\mathbf{u}^{*}) + H(\tilde{\mathbf{p}}) \). In other words, the phonon component and the dislocation component separate.

In the Fourier representation, the dislocation part of the displacement tensor \( \mathbf{a}_b(\mathbf{r}) \) is written

\[
\mathbf{a}_b(\mathbf{r}) = \frac{\mathbf{X}}{2\mathbf{a}} \frac{\partial}{\partial \mathbf{u}} \mathbf{u}^{*}(\mathbf{r}) - \frac{\mathbf{X}}{2\mathbf{a}} \frac{\partial}{\partial \mathbf{u}} \mathbf{u}^{*}(\mathbf{r}) \tilde{\mathbf{p}}(\mathbf{r}) + \mathbf{v}(\mathbf{r}) \tilde{\mathbf{p}}(\mathbf{r}),
\]

where \( \tilde{\mathbf{p}}(\mathbf{r}) \) are the Fourier transforms of respectively the dislocation density, the electron-impurity potential, and the impurity concentration.

The last term in (24), \( \frac{\partial}{\partial \mathbf{u}} \mathbf{u}^{*}(\mathbf{r}) \tilde{\mathbf{p}}(\mathbf{r}) \), describes the distortion of the electron lattice by the impurities which are introduced; the screening of the impurity fields is related to this distortion. Let us discuss this effect.

4.2. Screening of charged impurities in a 2D Wigner crystal

We first note that the random field corresponding to impurities with fairly long-range interactions are singular. For 6-correlated Coulomb impurities, for example (this is the case which we will be discussing below),

\[
\frac{\partial}{\partial \mathbf{u}} \mathbf{u}^{*}(\mathbf{r}) \tilde{\mathbf{p}}(\mathbf{r}) = \sum_{\mathbf{k}} \frac{\mathbf{Q}_k(\mathbf{r})}{2\mathbf{a}} \tilde{\mathbf{p}}(\mathbf{r}),
\]

diverges logarithmically at large distances in the 2D system. The mean square fluctuations of the potential at a point are proportional to the logarithm of the area of the system. There is accordingly the question of whether fields of this type can actually be achieved in real systems.\(^{59}\)

We note in this connection that the total field (the field of the impurities plus the field of the deformed lattice),

\[
\mathbf{q}_{\mathbf{k}}(\mathbf{r}) = \int \frac{d^2 q}{(2\pi)^2} \mathbf{G}(q) \tilde{\mathbf{p}}(\mathbf{r}) = \int \frac{d^2 q}{(2\pi)^2} \frac{\mathbf{Q}_k(\mathbf{r})}{2\mathbf{a}},
\]

which is the observable field, turns out to be regular, as can be seen from (27).

From (27) we also find an expression for the screened potential of an individual impurity of charge \( e \) at point \( \mathbf{r}_0 \):

\[
\mathbf{q}_{\mathbf{k}}(\mathbf{r}) = \sum_{\mathbf{R}} \frac{e}{|\mathbf{q}|} \tilde{\mathbf{p}}(\mathbf{r} + \mathbf{R}),
\]

The screening of a Coulomb center by the electrons of a 2D lattice thus occurs in qualitatively the same way (the potential varies as \( r^{-3} \) as \( |\mathbf{r}| \to \infty \)) as the screening by free 2D electrons (Ref. 27, for example).

4.3. Effective replica interaction of dislocations

Using (26) and (25), we find the following result for the dimensionless dislocation Hamiltonian \( H_b = H(\mathbf{u}^{*})/T \):

\[
H_b = \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \left[ \mathbf{A}_b \cdot \mathbf{A}_b \right] + \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \left[ \mathbf{a}_b \cdot \mathbf{a}_b \right].
\]

The part of Hamiltonian \( H_{\perp} \) in (28) which is quadratic in \( \mathbf{r} \), i.e., \( H_{\perp} \), can be put in the form of the potential energy of binary interactions and the sum of their "intrinsic" energies (we are thinking of configurations with \( \Xi, \mathbf{b}^2 = 0 \), which have finite energies in the thermodynamic limit):

\[
H_{\perp} = \frac{K}{8\pi} \int d^2 r \int d^2 r' \left( U_e(\mathbf{r} - \mathbf{r}') - U_\Xi(\mathbf{r} - \mathbf{r}') \right) + \frac{E_e}{2\pi} \int d^2 r \left\| \mathbf{r} \right\|,
\]

where \( K \) is a constant coupling constant, \( E_e \) is the energy of the disorder core, and \( a = a_0 \) is the size of this core.

The potential in (30) is broken up into the sum of (a) the usual binary-interaction potential for \( 2D \) dislocations, \( U_e(\mathbf{r}) = \delta_{\mathbf{r}, 0} \ln \left( \frac{\mathbf{r}}{a_0} \right) - \frac{2}{3\pi} \frac{1}{r^2} \),

\[
\mathbf{a}_b(\mathbf{r}) = \frac{e}{\mathbf{a}} \int \frac{d^2 q}{(2\pi)^2} \mathbf{G}(q) \tilde{\mathbf{p}}(\mathbf{r}),
\]

which corresponds to a 2D vector Coulomb gas,\(^{12},10\) and (b) an increment \( \Delta U_b(\mathbf{r}) \), which falls off by a power law (as \( r^{-1.5} \)). Physically, it is obvious that the increment \( \Delta U_b(\mathbf{r}) \) can be ignored in the phase-transition problem, in which the remote asymptotic behavior is important (indeed, this increment always is ignored in this case).

The role played by various impurities can be analyzed qualitatively by a similar approach. For this purpose we switch to an effective Hamiltonian in which an additional interaction arises after an average is taken over the random field between dislocations. The nature of this interaction is important to the phase-transition problem.

To average the logarithm of the partition function in the expression for the free energy over the frozen disorder, we use a replica method.\(^{59}\)

\[
F = -T \ln Z = -T \lim_{M \to \infty} \frac{\langle Z^M \rangle - 1}{M},
\]

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The quantity \( Z \) for \( Z = \sum_i \exp(-\beta H_i) \) has the form of the partition function of an \( M \)-component system of interacting replicas:

\[
Z_{\text{rep}} = \sum_{\{n_i\}} \exp(-\beta H_{\text{rep}}),
\]

(33)

where \( H_{\text{rep}} = \frac{1}{2} \sum_{i,j} n_i n_j K_{ij} \). The coefficient for the \( q \)-th component is

\[
K_q = \frac{1}{2} \sum_{i,j} n_i n_j K_{ij} \rightarrow K_{\text{rep}} - K_q \frac{\sum_{i} n_i^2}{\sum_{i} n_i},
\]

(34)

The trace over the distance variables is to be understood as the continuum expression (Ref. 13, for example)

\[
T_{ij} = \sum_{i,j} \left( \delta_{ij} \frac{1}{\sigma_i} \right)^2 \cdot \sum_{i,j} \delta_{ij} \cdot \sum_{i,j} \delta_{ij},
\]

(35)

The parameter \( \gamma = \exp(-E_0/T) \) gives the probability for finding distance-ions in the large system, \( N = \sum_{i,j} n_i \) is the total number of distances, \( n_i \) is the number of distances of unit length in configuration \( \{n_i\} \) with vector \( b \) directed along one of the six possible directions (for a triangular lattice with \( b = 1 \)). In the pure system, dislocations with \( b > 1 \) are unimportant from the renormalization-group standpoint. 227,28 The impurity case is discussed in Section 5.2.

For all impurities which conserve the quasilong-range translational order in the harmonic approximation, the coupling constant \( K_q \) remains finite as \( q \to 0 \). In the case \( K_q = q^2, K > 0 \), the interaction falls off as a power law at large distances and can be ignored, as in a pure system (there is no reason to expect the appearance of singularities as \( M \to 0 \)). The potential of the interaction of such impurities with the lattice falls off more rapidly than \( r^{-\alpha} \) as \( r \to 0 \) (Subsec. 3.3.1). For impurities with a Coulomb potential the coupling constant is

\[
K_q = K_q \rightarrow K_q = 16\pi \frac{a}{\sigma_i},
\]

(36)

and the additional interaction between dislocations (with a distinct temperature dependence \( T^{-\alpha} \) and the opposite sign), has the same functional form (see [22]) as that for the pure system.

The corresponding critical behavior (as \( M \to 0 \)) was studied in Ref. 16. Again in the phase-transition problem, Coulomb impurities in a 2D Wigner crystal thus play the role of short-range crystals for 2D crystals with a short-range impurities (see also Subsec. 3.3.2), and the results of Ref. 16 can be transferred to our case. The most important of these results will be summarized below in Sec. 5, where we also analyze the approximate Wilson renormalization-group equations corresponding to (37).

5. SCALE DIMENSIONS OF THE IMPURITY COUPLING CONSTANTS

5.1. To determine the role played by various impurities from the renormalization-group standpoint, we make use of the well-known duality between the grand canonical sum of a 2D Coulomb gas (with a logarithmic interaction) and the sine-Gordon model (Refs. 25-32, for example). To go over to the field model we write \( \exp(-\beta H) \) as the ratio of two functional integrals in terms of auxiliary two-component fields \( \Phi, \Phi' \), i.e., \( \exp(-\beta \Phi) \). After the trace is taken over the dislocation variables (Ref. 30, for example), \( Z_{\text{rep}} \) becomes

\[
Z_{\text{rep}} = \sum_{\{\Phi, \Phi'\}} \exp(-\beta \mathcal{L}(\Phi', \Phi)),
\]

(38)

where

\[
\mathcal{L}(\Phi, \Phi') = \frac{1}{2} \sum_{\substack{r, \Phi, \Phi' \in \{0, 1\} \\text{for} \\Phi \\text{and} \\Phi' \text{in the same coordinate axis in} \mathbb{R}^3.}} \exp(-\beta \Phi) \cdot \exp(-\beta \Phi'),
\]

(39)

and

\[
U = \int d\Phi d\Phi' \sum_{\Phi, \Phi' \in \mathcal{R}} \Phi \cdot \Phi' \cdot \exp(-\beta \mathcal{L}(\Phi', \Phi)),
\]

(40)

There is an upper cutoff momentum \( \Lambda \approx \pi^{-1} \) in (38) and (39); we have omitted the purely Gaussian normalization denominator. We have ignored the angular dependence of the interaction potential in (32); formally, we did this by making the replacement \( \int d\Phi \cdot \exp(-\beta \Phi) \rightarrow \int d\Phi \cdot \exp(-\beta \Phi) \cdot \cos \Phi \cdot \exp(-\beta \Phi) \). One can verify that, again in the case of impurities (which conserve the quasilong-range order), the angular terms are intermediate in the renormalization-group sense, as in the pure system. 227,28

For impurities with a Coulomb potential \( \alpha = r^{-1} \) we have

\[
\mathcal{L}(\Phi, \Phi') = \int d\Phi d\Phi' \sum_{\Phi, \Phi' \in \mathcal{R}} \exp(-\beta \Phi) \cdot \exp(-\beta \Phi'),
\]

(41)

in the limit \( \phi = 0 \). The increment \( \delta K \) is

\[
\delta K = K - K + 0 (\text{on} \theta, \Phi, \Phi'),
\]

(42)

The approximate Wilson renormalization-group equations take the following form as \( M \to 0 \) (see the Appendix):
our approach (see the Appendix). This position is the same as in Ref. 16.

For all other impurities which conserve the quasi-long-range translational order (for which the potential fall off more rapidly than $r^{-1}$), the coupling constant at small values of $q$ [as at $\Delta K \ll q$ (4)] is of the form $K_{\alpha} \propto q^{\omega} W_{\alpha} q^{\beta}$, where $\beta > 0$ (i.e., $n$). It can be shown easily in the usual way that the scale dimensions corresponding to $W_{\alpha}$ are negative in this case and equal to $-\beta$. The impurity coupling constants are inconsequential in the renormalization-group sense.27,34

5.2. When the system undergoes low-temperature reentrant melting, the effective repulsive interaction of a dislocation pair with opposite Burgers vectors becomes repulsive at low temperatures, $K < 0$ with $K^{-1} < \beta$. It may thus turn out that for pairs of coupled dislocations with $|b| > 1$ (which are unimportant in the absence of a disorder35), the "instability" occurs earlier than for a pair of dislocations of unit length. Let us examine this possibility. Dislocations with all possible Burgers vectors $b = n \mathbf{a}_1 - m \mathbf{a}_2$ make contributions of the form

$$
\sum_{n,m} \sum_{l,m} \sum_{\mathbf{R}} \delta R \cos (n \mathbf{a}_1 - m \mathbf{a}_2 \cdot \mathbf{R})
$$

(42)

to Hamiltonian (39). The scale dimensions of the coupling constants $v_{\alpha}$, $(q_{\alpha} a_\alpha) = 2a_\alpha^2$ are equal to $-2 - n^2 + m^2 + n m / (K \delta r)$. In the stability region, with $d y / d l > 0$, they turn out to be negative, and these charges become 'inert', and in principle all should be taken into consideration.

6. CONCLUSION

Taking account of the compressibility of a classical 2D Wigner crystal in the long-wave limit, we have analyzed the effect of a frozen-in disorder of two types on the dislocation melting of such a crystal. The first type of disorder consists of randomly distributed isotropic impurities with a variety of impurity-lattice interaction potentials, including long-range potentials. The second type of disorder was a random force with short-range correlations, stemming from irregularities of the substrate. Since the field of isotropic impurities interacts with only the hard longitudinal plasma modes of a 2D Wigner crystal, impurities whose potentials fall off more rapidly than $r^{-1}$ do not affect the dislocation melting (they are inconsequential from the standpoint of the critical behavior). Randomly distributed charged Coulomb impurities act as short-range impurities in 2D crystals with a short-range interaction.26 If the concentration of these impurities does not exceed the critical value, $\alpha_{\alpha} = n_{\alpha} / (\pi \mathbf{a}^2)$, the Coulomb impurities cause only a temperature shift of ordinary dislocation melting but also a reentrant low-temperature melting. As a result, the 2D Wigner crystal exists only in a bounded temperature interval. In the case of $n_{\alpha} > \alpha_{\alpha}$ fluctuations of the random impurity field break up the dislocation pairs, and a crystalline state with a quasi-long-range order is impossible (the translational order falls off exponentially). A situation of this sort can occur, for example, in the case of a 2D Wigner crystal in a heterostructure with a modulational doping, in which there is a high electron mobility (and there are also isolated dots of surface concentration $n_{\alpha} < \alpha_{\alpha}$ at a minimal distance $\delta = \alpha_{\alpha} / \alpha_{\alpha}$ from the plane of the 2D Wigner crystal, lying in a layer of finite thickness $1\alpha_{\alpha}$). At low temperatures, however, the regions of a short-range crystalline order may be large enough (at least no smaller than $2\alpha_{\alpha}$; see (19)) to be detected in magnetotransport experiments.

Even when harmonic phonons alone are taken into account, the random force $\alpha_{\alpha}$, which is added to the substrate (irregularities) disrupts the translational 2D order of the 2D Wigner crystal, converting this order into a short-range exponential order. In this system, which may be thought of as a Wigner glass with a quasi-long-range orientation order, a dislocation melting is not possible as a true critical behavior. If there is a slight disorder, however, and the dimensions of the crystalline domains are large in comparison with the lattice constant, the system is physically a Wigner crystal with a finite correlation length, and it can undergo some diffuse transitions involving the dissociation of dislocation pairs—either the ordinary transition or a low-temperature reentrant transition. We wish to thank S. M. Apenko for many stimulating discussions.

APPENDIX

Here we derive renormalization-group equations (41). We write the field $\Psi$ as the sum of long-wave and short-wave parts:

$$
\Psi(x) = \Psi(x) + \Psi'(x),
$$

$$
\Psi(x) = \int_{\xi \sim \xi_0} d^q \Psi \left( \xi \right) \psi \left( q \right),
$$

$$
\Psi'(x) = \int_{\xi \sim \xi_0} d^q \Psi' \left( \xi \right) \psi' \left( q \right),
$$

(41)

Here $x$ is related to an infinitesimal change of scale $\delta l$ by the relation $x = 1 + \delta l$. We will integrate in $\xi_0$ in $\Sigma_{\Psi}$ (38) at the short-wave fields $\Psi$. Here we use a perturbation theory in the small parameter $\psi \ll 1$. Within terms on the order of $\psi^2$ inclusively, we find

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where \( \langle \cdots \rangle \) means an average with quadratic Hamiltonian \( H_0(R_1, R_2) \) and \( \tilde{H} \) is given by (cf. Ref. 32)

\[
\tilde{H} = \hbar^2 \prod_{j=1}^{\infty} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2} \sum_{j=1}^{\infty} \left[ \frac{1}{4} \hbar^2 \frac{\partial^2}{\partial x_j^2} + \sum_{l=1}^{\infty} \left[ \hbar^2 \frac{\partial^2}{\partial y_j^2} - \frac{1}{\hbar^2} \frac{\partial^2}{\partial y_l^2} \right] \cos(\hbar x_j x_l - \hbar y_j y_l) \right]
\]

Here the propagator is

\[
h_{\text{e}}(x_j, y_l|y_j, y_l) \propto \exp \left( - \frac{\hbar^2}{2} x_j^2 \right)
\]

and \( J_0(x) \) is a Bessel function. To obtain the renormalization of the coupling constants, we would like to use a gradient expansion to put the part \( \tilde{H} \) in (A2) in the form of the original Hamiltonian. However, if we use expression (A3) for the propagator, \( G_{\text{e}}(R) \), the integrals which become infinite in \( R \) space. When the fields are cut off smoothly [in contrast with the sharp cut-off in (A1)] we could expect to divide \( G_{\text{e}}(R) \) to fall off fairly rapidly at distances \( R \sim \sigma^{-1} \). For a qualitative analysis of the critical behavior at low temperatures, we therefore refer to Ref. 35 (see also Ref. 36) in assuming that a smooth cut-off procedure is carried out. We moreover assume that this procedure leads to the result

\[
G_{\text{e}}(R) = \frac{N_0}{\hbar^2} \frac{J_0(\sqrt{2N_0} R)}{R}
\]

where the function \( J_0(x) \) falls off rapidly at \( x \rightarrow 0 \). The renormalization of the coupling constants \( \xi_{\text{e}} \) is given by the terms in Hamiltonian \( \tilde{H} \) which contain \( \cos(\hbar x_j x_l - \hbar y_j y_l) \) with \( j \neq l = n \). Using a gradient expansion to write them in the form

\[
\sum_{j \neq l \neq n \neq j} \left[ \hbar^2 \frac{\partial^2}{\partial x_j^2} - \frac{1}{\hbar^2} \frac{\partial^2}{\partial y_l^2} \right] \cos(\hbar x_j x_l - \hbar y_j y_l)
\]

and integrating over \( R = \rho - R' \), we find

\[
\frac{dS_{\text{e}}}{d\rho} = \frac{3}{4} \hbar^2 \frac{\partial^2}{\partial x_j^2} \sum_{j \neq l \neq n} \left[ \frac{1}{\hbar^2} \frac{\partial^2}{\partial y_l^2} \right] \cos(\hbar x_j x_l - \hbar y_j y_l)
\]

for a triangular lattice. The renormalization of \( \xi \) is given in first order in \( \rho \) by the first term in \( \tilde{H} \). In second order—this is an effect which occurs for a triangular lattice, and which formally results from \( \xi \rightarrow 0 \)—this renormalization can again be found through a gradient expansion from terms containing \( \cos(\hbar x_j x_l - \hbar y_j y_l) \) with \( j \neq l \neq n \). To restore the original scale we transform to the new fields \( \tilde{q} \) via \( \tilde{q} = \frac{1}{\delta} q \), where \( \delta = \frac{\hbar}{\delta} \). We choose the parameter \( \delta \) in this way so that a coordinate representation to \( \tilde{q} \) and \( \tilde{r} \) is a translation.

\[
\frac{dS_{\text{e}}}{d\rho} = \frac{3}{4} \hbar^2 \delta \frac{\partial^2}{\partial x_j^2} \sum_{j \neq l \neq n} \left[ \frac{1}{\hbar^2} \frac{\partial^2}{\partial y_l^2} \right] \cos(\hbar x_j x_l - \hbar y_j y_l)
\]

where \( \delta = \delta x_{\text{e}} \rho_{\text{e}}(x) \). In the limit \( M \rightarrow 0 \) we find (41), making use of the circumstance that \( \frac{dS_{\text{e}}}{d\rho} \) is diagonal with respect to the replica indices.

1) It corresponds to the incorporation in (28) of a dependence of the coupling constant on the wave vector \( \mathbf{q} \) in \( \tilde{K}(\mathbf{q}) = K(\mathbf{q}) \), where \( K(\mathbf{q}) = K + (\mathbf{q}^2)/(2r) \), and \( \tilde{K}(\mathbf{q}) \). A corresponding increase arises when the dependence of the elastic constants \( \psi(q) \) and \( \mu(q) \) is taken into account.

2) A procedure used in Ref. 32 to regularize the propagator \( G_{\text{e}}(R) \) is to effect a smooth cut-off. This procedure consists of replacing \( \tilde{q} \) by \( \tilde{q} - \alpha \tilde{q} \) in the propagator and switching to an integration over all momentum space. Some essential or divergence arise in the process and render this model unrenormalizable at low temperatures, \( K < 1/(4\pi) \). Furthermore, the model itself can be defined only as a double expansion in the parameters \( x_0 \) and \( K > 1/(4\pi) \) (Ref. 29). For this reason, the approximate approach of Ref. 32 works only near \( K > 16e^{-1/8} \).


