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A CLASS OF EXACTLY SOLVABLE MANY-PARTICLE MODELS

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A class of many-particle quantum models with arbitrary space dimension and arbitrary particle statistics for which it is possible to construct a number of exact eigenstates is found. In the models, the following assumptions are made: 1) the presence of two (or 2m) components with "symmetric" matrix elements of the two-body interactions (equality up to the sign of all the interaction potentials and equality up to a phase factor of the wave functions of particles of two species; otherwise the two-body interactions are arbitrary), 2) degeneracy of the (total) spectrum of the free particles. The exact states correspond to a condensate of noninteracting composite particles ("excitons") that are not precisely bosons and to excitations over the condensate. The possibility of exact solution rests on the symmetry with respect to continuous rotations in the isospin space, this corresponding to Bogolyubov transformations with momentum-independent parameters u, v. The class includes, in particular, two-dimensional electron–hole systems in a strong magnetic field.

1. Introduction

Comparatively few quantum many-body problems that admit exact solution are known. As a rule they correspond to the one-dimensional case, or to problems with short-range two-body potentials, or to problems that admit a solution that is asymptotically exact in the thermodynamic limit (see [1–4] and the literature quoted there).

It turns out that there exists a class of models of arbitrary space dimension for which one can construct a number of exact eigenstates for two-body potentials of arbitrary form, including long-range potentials. The important common properties of these models are: 1) the presence of two (or 2m) components with "symmetric" interaction matrix elements, 2) degeneracy of the total spectrum of the free particles.

Models of this kind arise, for example, when one considers a system of two-dimensional (2D) electrons (e) and holes (h) in a sufficiently strong magnetic field. In this case, the virtual transitions between the different Landau levels are negligibly small and one can restrict consideration to only the partly filled (in the simplest case, zeroth) macroscopically degenerate Landau levels of the e and h.

In [5], the many-particle effects in the ground state of this system were analyzed systematically by the methods of a temperature-dependent diagram technique including anomalous exciton pairings in the zeroth approximation with subsequent passage to the limit $T \to 0$ (because of the macroscopic degeneracy, the usual perturbative approaches are not valid for $T = 0$, and in the limit $T \to 0$ the temperature-dependent diagram technique gives divergences). It was found that the ground-state energy of the 2D system in a given Landau level can be calculated exactly and is equal to the sum of the binding energies of 2D magnetic excitons of zero momentum [6]. A direct quantum-mechanical treatment made it possible to construct the ground-state wave function in the form of an exciton condensate, and also some excited states [7] (the ground-state wave function was also obtained independently in [8], where, however, its relation to an exciton condensate was not established explicitly).

Problems that are exactly solvable were also found: a multicomponent layered system of two-dimensional e and h in a strong magnetic field, corresponding to several quantum wells; a system in crossed electric and strong magnetic fields [9].

The last situation is interesting because of the basic possibility [9] of

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dissipationless transport of excitation energy by nonequilibrium excitons, which can be regarded as an analog of the quantum Hall effect for the case of a neutral two-component system. A very similar treatment was given subsequently in [10].

Equivalent models also arise when one considers 2D electron systems in a strong magnetic field with several equivalent groups of electrons — in many-valley semiconductors [11], in the description of spin excitations that arise on the transition of an electron to a Landau level with the same number but opposite spin projections (for the dispersion laws of such excitations, see [12,13]).

Here, we wish to identify the general properties of the Hamiltonians of such systems that make it possible to construct for them exact many-particle states and give a very simple derivation based on the operator algebra.

2. The Hamiltonian

We consider a system containing two particle species. The Hamiltonian of the free particles has the form

$$H_0=\sum_{i=1,2}\sum_{p}\varepsilon_i(p)a_{ip}{}^+a_{ip}, \tag{1}$$

where $a_{1p}{}^+$, $a_{2p}{}^+$ are the Fermi operators of creation of particles of type 1 and 2 with (quasi)-continuous quantum number p, the number of components of which need not necessarily be equal to the dimension D of space. Without loss of generality, it can be regarded as a (quasi)momentum taking N_0 different values (for e and h on one Landau level, N_0 is the degeneracy of the level).

Let the dispersion laws of the components satisfy the condition

$$\varepsilon_1(p)+\varepsilon_2(-p)=\varepsilon_0, \tag{2}$$

where ε_0 is a constant that does not depend on p (for a semiconductor, ε_0 is the width of the forbidden band). We emphasize that we use the hole description of one group of carriers (say, with the number 2). If they have an electron description $\varepsilon_2(p)\to-\varepsilon_2(-p)$, $a_{2p}{}^+\rightleftarrows a_{2,-p}$ and (see below) $U_{12}(\mathbf{r})\to-U_{12}(\mathbf{r})$, $\Phi_{2p}(\mathbf{r})\to\Phi^*_{2,-p}(\mathbf{r})$.

The Hamiltonian of the two-body interactions is given by the expression

$$H_{int}=\frac{1}{2}\sum_{i,j=1,2}\sum_{p_1,\ldots,p_2'}U_{ij}(p_1,p_2;p_1',p_2')a^+_{ip_1}a^+_{jp_2}a_{jp_2'}a_{ip_1'}. \tag{3}$$

We assume that the interaction matrix elements have the properties

$$U_{ij}(p_1,p_2;p_1',p_2')=U_{ji}(p_2,p_1;p_2',p_1'). \tag{4}$$

$$U_{11}(p_1,p_2;p_1',p_2')=-U_{12}(p_1,-p_2';p_1',-p_2)=U_{22}(-p_1',-p_2';-p_1,-p_2), \tag{5}$$

$$U_{11}(p_1,p_2;p_1',p_2')=\delta_{p_1+p_2,\,p_1'+p_2'}V(p_1,p_1',p_2'), \tag{6}$$

$$V(p_1,p_1',p_2')=v(p_1-p_1',p_1-p_2'). \tag{7}$$

To clarify what corresponds to the conditions (4)-(7), we go over to the coordinate representation. Let $\Phi_{ip}(\mathbf{r})$ be a wave function of the free particles, $U_{ij}(\mathbf{r})$ be the potentials of the two-body interaction ($U_{12}(\mathbf{r})=U_{21}(\mathbf{r})$ by definition), and the interaction matrix elements be defined in the usual manner as

$$U_{ij}(p_1,p_2;p_1',p_2')=\int d\mathbf{r}_1\int d\mathbf{r}_2\Phi_{ip_1}{}^*(\mathbf{r}_1)\Phi_{jp_2}{}^*(\mathbf{r}_2)\,U_{ij}(\mathbf{r}_1-\mathbf{r}_2)\Phi_{jp_2'}(\mathbf{r}_2)\Phi_{ip_1'}(\mathbf{r}_1).$$

Then (4) corresponds to $U_{ij}(\mathbf{r})=U_{ij}(-\mathbf{r})$, which is a general property of two-body potentials, and the condition (5) to fulfillment of the requirements

$$U_{11}(\mathbf{r})=U_{22}(\mathbf{r})=-U_{12}(\mathbf{r})=U(\mathbf{r}), \tag{8}$$

$$\Phi_{1p}{}^*(\mathbf{r})\Phi_{1p'}(\mathbf{r})=\Phi_{2,-p}(\mathbf{r})\Phi^*_{2,-p'}(\mathbf{r}), \tag{9}$$

i.e., between identical particles there is identical repulsion, between particles of different species there is the same attraction, and their wave functions in the electron representation are identical up to a phase factor. In addition, there is the conservation law (6), and the matrix element (7) depends only on differences of arguments: the momentum transfer $p_1 - p_1'$ and the "difference" momentum $p_1 - p_2'$ (for plane wave $\Phi_{ip}(\mathbf{r})$, $v(p_1 - p_1', p_1 - p_2')$ reduces to the Fourier transform of the interaction potential $\tilde{U}(p_1 - p_1')$).

3. Condensate of Composite Particles as Exact State

It can be shown that when the conditions (2) and (4)–(7) are satisfied, the total Hamiltonian $H = H_0 + H_{int}$ admits the construction of exact many-particle eigenstates. Indeed, let us consider the normalized operator of creation of a composite particle

$$Q_0^+ = \frac{1}{\sqrt{N_0}} \sum_p a_{2p}^+ a_{1,-p}^+ \tag{10}$$

(as will be seen, Q_0^+ corresponds to a pair bound state that is densest in the $\mathbf{r}$ space). Quite unexpectedly, the exact quantum equation of motion for Q_0^+ has the form of a finite operator algebra:

$$[H, Q_0^+] = \varepsilon Q_0^+, \tag{11}$$

$$\varepsilon = \varepsilon_0 + E_0, \quad E_0 = -\sum_p v(p, 0), \tag{12}$$

where ε_0 and $v(p, p')$ are determined in (2) and (7), respectively. It follows immediately from (11) that

$$H(Q_0^+)^N|0\rangle = N\varepsilon(Q_0^+)^N|0\rangle, \tag{13}$$

where $|0\rangle$ is the vacuum state. Thus, the state with condensate of composite particles $(Q_0^+)^N|0\rangle$ is an exact eigenstate of the many-particle Hamiltonian H. (The question of whether this state of the many-particle system is the ground state, as in the case of a 2D e–h system in a strong magnetic field, can be obtained only by investigating a specific physical realization — see Sec. 6.2.)

It follows from Eq. (11) that the particles Q_0^+ form an ideal gas — they interact neither with one another nor with arbitrary excitations over the condensate. Indeed, suppose that such an excitation corresponds to an operator f^+ containing an arbitrary combination of creation operators a_{ip}^+ and such that $Hf^+|0\rangle = E_f f^+|0\rangle$. Then for the state $f^+(Q_0^+)^N|0\rangle$, taking into account the commutation relation $[f^+, Q_0^+] = 0$, we obtain from (13)

$$Hf^+(Q_0^+)^N|0\rangle = (N\varepsilon + E_f)f^+(Q_0^+)^N|0\rangle. \tag{14}$$

We give some examples below in Sec. 6.

4. Exact Symmetry of the Hamiltonian

The symmetry on which the relation (11) is based is isospin symmetry of the components:

$$a_{1,p} \rightleftarrows a_{2,-p}^+.$$

It was used explicitly in [7] to construct the ground-state wave function of a 2D e–h system in the zeroth Landau level. Two-dimensional many-valley semiconductors in a strong magnetic field have a similar symmetry [11].

The presence of this symmetry can be demonstrated by using Bogolyubov transformations [14]. By their means, as is well known, one can exactly diagonalize an arbitrary form quadratic in the operators (see, for example, [15]).

We intend to show that for the considered class of Hamiltonians with two-body interaction there exist Bogolyubov transformations that make it possible to find many-particle eigenstates.

Namely, we consider transformations corresponding to exciton pairings with transformation parameters u, v that do not depend on the continuous quantum number p. For the case of Fermi statistics, they have the form

$$a_{1,p} \to \tilde{a}_{1,p} = ua_{1,p} + va_{2,-p}^+, \quad a_{2,p} \to \tilde{a}_{2,p} = ua_{2,p} - va_{1,-p}^+, \tag{15}$$

where $u^2 + v^2 = 1$. The transformations (15) are realized by a unitary operator that in the given case has the form (see, for example, [16])

$$S=\exp\{\Theta(Q_0^+-Q_0)\}, \tag{16}$$

so that $\tilde{a}_{ip}=Sa_{ip}S^+$, where $u=\cos(\Theta/\sqrt{N_0})$, $v=\sin(\Theta/\sqrt{N_0})$.

It is very important — and the basis of the possibility of exact solution — that the anti-Hermitian generator of rotations $\hat{L} = Q_0^+ - Q_0$ is directly related to the creation operator Q_0^+ of the composite particle.

Since the transformations (15) do not preserve the particle number, it is convenient to go over in the usual manner to the Hamiltonian

$$\hat{H}=H-\mu\hat{N}, \quad \hat{N}=\hat{N}_1+\hat{N}_2=\sum_i\sum_p a_{ip}{}^+a_{ip}, \tag{17}$$

where μ are the equal chemical potentials of the components.

The transformed Hamiltonian has the form [7]

$$S\hat{H}S^+=W(\Theta)+\bar{H}_0+\bar{H}_{\rm int}, \tag{18}$$

where $W = (\varepsilon - 2\mu)v^2N_0$ is a numerical function of the parameter Θ,

$$\bar{H}_0=-uv(\varepsilon-2\mu)\sum_p(a_{2p}{}^+a_{1,-p}^+ +\text{h.c.})-[v^2(\varepsilon-2\mu)+\mu]\hat{N} \tag{19}$$

is the two-particle part of the Hamiltonian, and $\bar{H}_{int} = H_{int}$, which retains its form (3), is the Hamiltonian of the two-body interactions.

If we set $\mu = \mu_0 = \varepsilon/2$, which corresponds to "Bose" condensation of the composite particles in a state with energy $2\mu_0 = \varepsilon$, then it can be shown that the Hamiltonian $\hat{H}$ (17) is exactly invariant with respect to rotations:

$$S\hat{H}S^+=\hat{H}. \tag{20}$$

We explain this rather unexpected result. An important property of the transformations (15) with constant coefficients u, v is that in the interaction Hamiltonian of the symmetric (see (8), (9)) two-component system (the actual form of the interaction potential is unimportant) nondiagonal fourfold terms of the form $a_1{}^+a_2{}^+a_1{}^+a_2{}^+$, $a_1{}^+a_2{}^+a_1{}^+a_1$, $a_2{}^+a_1{}^+a_2{}^+a_2$ and their Hermitian conjugates do not appear — see (18) (see, for example, [16]). In addition, a specific feature of the considered system is that by virtue of the condition (2) the coefficient of the two-body nondiagonal terms that arise in (19) is the constant $(\varepsilon - 2\mu)$, which can be made to vanish by the choice of the chemical potential μ.

It follows from (20) that the generator of rotations $\hat{L}$ is an integral of the motion: $[\hat{L}, \hat{H}] = 0$, whence $[\hat{H}, Q_0^+] = 0$ and

$$[H, Q_0{}^+]=\mu_0[\hat{N}, Q_0{}^+]=2\mu_0Q_0{}^+,$$

in exact agreement with the equation of motion (11).

We note one further interesting consequence of the invariance (20), from which there follows the exact relation

$$HS^+|0\rangle=\mu_0\hat{N}S^+|0\rangle. \tag{21}$$

Thus

$$S^+|0\rangle=\prod_p(u-va_{2p}{}^+a_{1,-p}^+)|0\rangle,$$

which is a state of BCS–Bogolyubov type [17,14] (it does not have a definite particle number and in the given case describes a coherent state of excitons), is, for the considered class of Hamiltonians, an exact state (an eigenstate of $\hat{H}$ with eigenvalue equal to zero).

Applying to both sides of (21) the operator $\hat{P}_{N,N}$ of projection onto states with $N_1 = N_2 = N$ particles, and taking into account the commutation relations

$$[H, \hat{P}_{N,N}]=[\hat{N}, \hat{P}_{N,N}]=0$$

(H and $\hat{N}$ preserve the particle number), we obtain in accordance with (13) an eigenstate of H with definite particle number:

$$H(\hat{P}_{N,N}S^+)|0\rangle=2\mu_0 N(\hat{P}_{N,N}S^+)|0\rangle, \quad (\hat{P}_{N,N}S^+)|0\rangle=\mathrm{const}(Q_0^+)^N|0\rangle.$$

5. Arbitrary Statistics of Components

It can be shown that the quantum equation of motion (11) remains the same when the statistics of one or both components is changed from the Fermi to the Bose case. This can be regarded as the system's having an additional discrete symmetry (see also [18]).

Because of this, the particles Q_0^+ correspond to an ideal gas irrespective of the statistics of the components, and Eq. (13) remains valid (at least formally). However, it must be emphasized that: 1) the composite particles Q_0^+, which are not precisely bosons, can condense in one quantum state, 2) the possible number N of particles in the condensate $(Q_0^+)^N|0\rangle$ is related to the statistics of the components.

The commutation relations satisfied by the operators Q_0^+ have the form

$$[Q_0^+,Q_0^+]=0, \quad [Q_0,Q_0^+]=1-\frac{\hat{N}_1+\hat{N}_2}{N_0}, \quad a_{ip} \text{ fermions}, \tag{22}$$

$$[Q_0^+,Q_0^+]=0, \quad [Q_0,Q_0^+]=1+\frac{\hat{N}_1+\hat{N}_2}{N_0}, \quad a_{ip} \text{ bosons}, \tag{23}$$

$$[Q_0^+,Q_0^+]_+=0, \quad [Q_0,Q_0^+]_+=1-\frac{\hat{N}_1-\hat{N}_2}{N_0}, \quad \begin{cases} a_{1p} \text{ fermions}, \\ a_{2p} \text{ bosons}, \end{cases} \tag{24}$$

where $\hat{N}_i$ are the particle-number operators, and $[\,,]_+$ denotes the anticommutator.

As follows from (24), the Q_0^+ particles with half-integer spin, which are not, strictly speaking, fermions, nevertheless satisfy the Pauli principle: Their possible population numbers are N = 0, 1. The composite particles with integer spin ((22), (23)) can, as is well known for excitons [16], be regarded as Bose formations only in the limit of low densities $N_i \ll N_0$. For Fermi statistics (22) of the components when $N_0 - N_i \ll N_0$ anti-excitons are Bose formations; the transition to them corresponds to the substitution $Q_0^+ \rightleftarrows Q_0$.

In the case (22), a restriction on the possible number of particles in the condensate follows from the Pauli principle: $N \leq N_0$. In the case (23) of Bose statistics of the components, N is obviously arbitrary. These assertions also follow from the values for the square of the norm of the state with condensate:

$$\langle 0|(Q_0)^N(Q_0^+)^N|0\rangle=\begin{cases} N!\left[\dfrac{N_0!}{N_0^N(N_0-N)!}\right], & N\leqslant N_0, \\ & a_{ip} \text{ fermions}, \\ 0, & N>N_0, \end{cases} \tag{25}$$

$$\langle 0|(Q_0)^N(Q_0^+)^N|0\rangle=N!\left[\frac{(N_0+N-1)!}{N_0^N(N_0-1)!}\right], \quad a_{ip} \text{ bosons}. \tag{26}$$

The factors in the square brackets in (25) and (26) arise because the particles do not have purely Bose statistics.

6. Some Examples

6.1. System of Particles with Infinite Mass. We consider a hypothetical system with arbitrary space dimension D containing two particle species for which the kinetic energy is absent, the (effective) masses being infinite. The Hamiltonian of the system reduces to the interaction Hamiltonian H_{int} (3), in which the operators a_{ip}^+ correspond to plane waves:

$$\Phi_{1p}(\mathbf{r})=\Phi_{2p}(\mathbf{r})=L^{-D/2}\exp(i\mathbf{pr}),$$

where $\mathbf{p}$ is the D-component momentum, and L^D is the volume of the normalization box.

We consider a two-particle problem, with one particle of each species. The Hamiltonian $H_{int}=U_{12}(\mathbf{r}_1-\mathbf{r}_2)$ commutes with the operators of the coordinate of the relative motion, $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, and of the center of mass, $\mathbf{R}=(\mathbf{r}_1-\mathbf{r}_2)/2$ (for simplicity, we set $m_1/m_2 = 1$), and also with the operator of the center-of-mass momentum: $\hat{\mathbf{P}}=-i\nabla_1-i\nabla_2$. We shall characterize the state of the pair by the values of the simultaneously observable $\mathbf{r}$ and $\hat{\mathbf{P}}$ ($[\mathbf{r},\hat{\mathbf{P}}]=0$). The wave function of the pair with definite values of $\mathbf{r}_0$ and $\mathbf{P}$ in the coordinate representation has the form

$$\Psi_{\mathbf{P}\mathbf{r}_0}(\mathbf{r}_1, \mathbf{r}_2) = L^{-D/2}\exp(i\mathbf{P}\mathbf{R})\delta(\mathbf{r}-\mathbf{r}_0). \quad (27)$$

For the wave function (27), we obviously have

$$H_{\rm int}\Psi_{\mathbf{P}\mathbf{r}_0}(\mathbf{r}_1, \mathbf{r}_2) = U_{12}(\mathbf{r}_0)\Psi_{\mathbf{P}\mathbf{r}_0}(\mathbf{r}_1, \mathbf{r}_2), \quad (28)$$

and the energy of the pair is degenerate with respect to $\mathbf{P}$.

In the second-quantization representation, the wave function (27) corresponds to the creation operator of the composite particle:

$$Q^+_{\mathbf{P}\mathbf{r}_0} = \sum_{\mathbf{p}} \exp(i\mathbf{p}\mathbf{r}_0)\, a^+_{2,\mathbf{P}/2+\mathbf{p}}\, a^+_{1,\mathbf{P}/2-\mathbf{p}}.$$

It cannot be normalized, as in (10), since $N_0 = \infty$, unless an ultraviolet regularization is made, for example, a cutoff at the momenta corresponding to the reciprocal lattice period.

For the operator Q^+_{00} of the most dense state with $\mathbf{r}_0=0$ and $\mathbf{P}=0$, we have the equation of motion (11), where from (12) in accordance with (28)

$$E_0 = -\int \frac{d^D q}{(2\pi)^D}\tilde{U}(q) = U_{12}(\mathbf{r}=0),$$

where $\tilde{U}(q)$ is the Fourier transform of the interaction potential (8).

Now $Q^+_{00}|0\rangle$ corresponds to a state in which a collapse of the e–h pair has occurred. This is possible, since for infinite masses there is no kinetic energy and the infinite uncertainty in the momentum of the relative motion is unimportant. In this sense, the model can be regarded as classical (cf. Sec. 6.2!). However, we emphasize that in the model the quantum exchange effects are taken into account exactly.

In accordance with (13) and (14), we find that the states with condensate $(Q^+_{00})^N|0\rangle$ and with one particle above the condensate, $Q_{\mathbf{P}\mathbf{r}}{}^+(Q_{00}{}^+)^{N-1}|0\rangle$, are exact eigenstates of the Hamiltonian. Note that for $r_0 = 0$ and $\mathbf{P}\neq 0$ no excitation energy is associated with removal of a particle from the condensate — the states $(Q^+_{00})^N|0\rangle$ are strongly degenerate.

6.2. System of Two-Dimensional Particles in the Lowest Landau Levels. We consider a 2D two-component e–h system in a strong magnetic field. We ignore the virtual transitions of the particles between the macroscopically degenerate Landau levels (the degeneracy is $N_0 = L^2/2\pi r_H{}^2$, $r_H = (\hbar c/eH)^{1/2}$ is the magnetic length). This is justified [6] when $r_H \ll a_B$, where a_B is the scale of the bound state in the potential of the interparticle interaction (for the Coulomb interaction, $a_B = \varepsilon\hbar^2/me^2$).

The wave functions of the particles in the zeroth Landau levels have the form

$$\Phi_{i p_y}(\mathbf{r}) = \frac{1}{(L\sqrt{\pi}r_H)^{1/2}}\exp\{ip_y y - (x \pm p_y r_H{}^2)^2/2r_H{}^2\}, \quad (29)$$

where p_y is the quantum number that determines the x coordinate of the center X of the e–h cyclotron orbit by $X = \pm p_y r_H^2$ (the gauge of the vector potential is $\mathbf{A} = (0, Hx, 0)$); X can take N_0 different values.

The state of the e–h pair is characterized by an exact quantum number — the 2D magnetic momentum $\mathbf{P}$ [19,20], which determines the mean distance between the particles in the pair. If mixing between the Landau levels is ignored, it does not depend on the form of the two-body potential and is given by the relation $\langle \mathbf{r}_{eh}\rangle = [\mathbf{e}_z\times\mathbf{P}]r_H{}^2$, $\mathbf{e}_z = \mathbf{H}/H$.

The second-quantization operator of the 2D magnetic exciton, found from the wave function in the coordinate representation [6], has the form [7]

$$Q_{\mathbf{P}}{}^+ = \frac{1}{\sqrt{N_0}}\sum_{p_y}\exp(iP_x p_y r_H{}^2)\, a^+_{2,P_y/2+p_y}\, a^+_{1,P_y/2-p_y}, \quad (30)$$

where $a^+_{ip_y}$ correspond to the wave functions (29).

The dispersion law $E(\mathbf{P})$ is determined by the relation

$$E(\mathbf{P})=\int\frac{d^2q}{(2\pi)^2}U(q)\exp(i\mathbf{q}\mathbf{P}r_H^2-q^2r_H^2/2). \tag{31}$$

For the creation operator of a 2D exciton of zero momentum, Q_0^+, we have the equation of motion (11), so that a state with condensate $(Q_0^+)^N|0\rangle$ is an exact eigenstate of the Hamiltonian. In accordance with (14), the following problems, for example, are also exactly solvable:

1) with one charge above the condensate of excitons:

$$f^+=a^+_{ip_y},\quad E_f=\varepsilon_i(p_y),$$

i.e., the excess charge does not polarize the exciton condensate;

2) with e–h pair that forms an exciton above the condensate with momentum P,

$$f^+=Q_{\mathbf{P}}^+,\quad E_f=E(\mathbf{P}),$$

which gives a spectrum of two-particle excitations that is determined by the exciton dispersion law $E(\mathbf{P})$; for any isotropic interaction potential $U(\mathbf{r})$, it is quadratic at small $\mathbf{P}$;

3) with three particles of light charges above the exciton condensate forming, despite the repulsive nature of the interaction, bound states with discrete spectrum in the strong magnetic field (see [12,21]).

Physically, the fact that the gas of 2D magnetic excitons is ideal can be explained as follows. Since the probability densities for the wave functions of the particles of the two components are equal, Eq. (9), the zero-momentum excitons Q_0^+ are truly neutral particles — they do not have dipole or any higher multipole moments. In addition, by virtue of momentum conservation transitions to states with $\mathbf{P}\neq 0$, which possess a dipole moment, are forbidden. Thus, if virtual transitions to higher Landau levels are ignored, there are no polarization interactions.

The exact equation of motion (11) also shows that in the system there are no exchange interactions between the zero-momentum excitons and the particles above the condensate resulting from overlapping of the wave functions of identical fermions. In the given case, it is simpler to obtain this result formally than to find a simple qualitative explanation of it. The canceling of the exchange interactions can probably be most readily traced for the example of the exciton + electron state $a^+_{1p_y}Q_0^+|0\rangle$. For this, we write it in the coordinate representation and consider the matrix elements of the total interaction Hamiltonian. Under the condition (9) and for the same (apart from the sign) interaction potentials (8) the corrections to the e–e and e–h interactions which arise on antisymmetrization of the total wave function with respect to the electron coordinates exactly compensate each other.

For real quasitwo-dimensional e–h systems in a strong magnetic field transitions between the Landau levels are important, and so is the possible asymmetry between e and h — differences in the interaction potentials and in the wave functions of the components. Qualitatively different effects are associated with them — virtual transitions to the higher Landau levels make the exciton gas weakly nonideal with repulsion between the particles, which stabilizes the exciton phase [5]; in the presence of e–h asymmetry, there is attraction between the excitons, and the exciton phase becomes absolutely unstable with respect to transition to a state with an e–h fluid [22] (see also [10] and the literature quoted there). Thus, the form of the ground state in a real system (exciton condensate or e–h fluid) is determined by the relative contribution of these two factors and can change when the magnetic field is increased, since this suppresses the contribution of the virtual transitions to the higher Landau levels.

6.3. Two-Dimensional e and h in Crossed Fields. We consider a system of 2D e and h in zeroth Landau levels in crossed fields $\boldsymbol{\mathscr{E}}\perp\mathbf{H}$ ($\boldsymbol{\mathscr{E}}=(\mathscr{E},0,0)$, the gauge is $\mathbf{A}=(0,Hx,0)$). We shall here find it convenient to introduce the following wave function system of the charged particles in the crossed fields [9]:

$$\tilde{\Phi}_{ip_y}(\mathbf{r})=\exp(im_i\mathbf{V}_0\mathbf{r})\Phi_{ip_y}(\mathbf{r}), \tag{32}$$

which satisfies (9) and differs from the wave function (26) in the absence of an electric field by the phase factor, in which $\mathbf{V}_0=c[\boldsymbol{\mathscr{E}}\times\mathbf{H}]/H^2$ is the drift velocity in the crossed fields (it does not depend on the sign of the charge), and m_i are the masses of the particles. This can be regarded as the result of Galilean transformations of the wave functions and energies (see (33) below) on the transition from the frame of reference in which $V_0=0$. For such a choice of the wave function, the quantum number p_y (1D momentum, measured from the drift momentum) describes as before the x coordinate of the center of the cyclotron motion.

The electric field lifts the degeneracy with respect to the position of the center of the cyclotron orbit, and the dispersion laws of the components take the form

$$\varepsilon_i(p_y)=\varepsilon_i^0-V_0p_y.$$

The operators of creation of excitons in the zeroth Landau level with momentum $\mathbf{P}$ in the crossed fields have the form [9]

$$\tilde{Q}_{\mathbf{P}}^+=\frac{1}{\sqrt{N_0}}\sum_{p_y}\exp(iP_x'p_yr_H^2)\tilde{a}^+_{2,P_y'/2+p_y}\tilde{a}^+_{1,P_y'/2-p_y},$$

where $\tilde{a}^+_{ip_y}$ corresponds to the wave function (32), and the momentum $\mathbf{P}'=\mathbf{P}-M\mathbf{V}_0$, $M=m_1+m_2$. For the exciton state, we have

$$H\tilde{Q}_{\mathbf{P}}^+|0\rangle=\tilde{\varepsilon}(\mathbf{P})\tilde{Q}_{\mathbf{P}}^+|0\rangle,\quad \tilde{\varepsilon}(\mathbf{P})=\varepsilon_0+M\mathbf{V}_0^2/2+\mathbf{V}_0\mathbf{P}'+E(P'), \tag{33}$$

where E(P) is given by (31). One can assume that the exciton energy is determined by the energy ε_0 of the particles in the lowest Landau levels, by the kinetic energy of the drift in the crossed fields, $MV_0^2/2$, by the energy of the interaction of the exciton dipole moment $\mathbf{d}=-e[\mathbf{e}_z\times\mathbf{P}']r_H^2$ with the field $\mathscr{E}$ (since $\mathbf{V}_0\mathbf{P}'=-\mathbf{d}\,\mathscr{E}$), and by the contribution of the interparticle interactions $E(P')$.

The densest pair state corresponds to an exciton with drift momentum $\mathbf{P}=M\mathbf{V}_0$ ($\mathbf{P}'=0$), for which the creation operator $\tilde{Q}_{\mathbf{P}_0}^+$ satisfies the equation of motion (11). Therefore the exciton condensate state $(Q_{\mathbf{P}_0}^+)^N|0\rangle$ is an exact eigenstate of the total Hamiltonian with eigenvalue $N(E_0+MV_0^2/2)$. It describes a drift of all particles with velocity $\mathbf{V}_0$. The state with condensate is unstable with respect to production of excitons above the condensate. Indeed, to the exact eigenstate with one exciton above the condensate, $\tilde{Q}_{\mathbf{P}}^+(\tilde{Q}_{\mathbf{P}_0}^+)^{N-1}|0\rangle$, there corresponds the energy $\delta\varepsilon=\mathbf{V}_0\delta\mathbf{P}+E(\delta P)$ (measured from the condensate energy), where $\delta\mathbf{P}=\mathbf{P}-\mathbf{P}_0$. Since $E(\delta P)$ for small δP is a quadratic function, $\delta\varepsilon$ does not have a minimum at $\delta\mathbf{P}=0$.

The situation is changed if allowance is made for transitions to the higher Landau levels. Then the excitation spectrum at low momenta becomes acoustic [5] $\delta\varepsilon\to\mathbf{V}_0\delta\mathbf{P}+c_{ac}\delta P+E(\delta P)$, and for $c_{ac}>V_0$ the state with condensate is stable with respect to the production of excitations [10]. Then in a nonequilibrium 2D e–h system a regime of almost dissipationless exciton flow can be realized [9,10].

6.4. Systems with 2m Components, Interaction with an External Field. We consider a system containing m two-component subsystems, which we shall denote by $\alpha = 1, \dots, m$. Such a model (for the case of Fermi statistics) corresponds, for example, to a set of m spatially separated quantum wells in a strong magnetic field; each contains two-dimensional e and h.

For each α, we assume that the dispersion laws of the components satisfy the condition $\varepsilon_1^{(\alpha)}(p)+\varepsilon_2^{(\alpha)}(-p)=\varepsilon_0^{(\alpha)}$. The Hamiltonian, taking into account the interactions H_{int} of all the particles, now has the form

$$H_{\rm int}=\sum_\alpha H_{\rm int}^{(\alpha)}+\sum_{\alpha<\beta}H_{\rm int}^{(\alpha\beta)},$$

where $H_{\rm int}^{(\alpha)}$ has the form (3) (with additional index α of the operators $a_{\alpha ip}$ and of the matrix elements $U_{\alpha i,\alpha j}$), and the Hamiltonian of the interaction of the particles of different "species," $H_{\rm int}^{(\alpha\beta)}$, is given by

$$H_{\rm int}^{(\alpha\beta)}=\sum_{i,j=1,2}\sum_{p_1,\dots}U_{\alpha i,\beta j}(p_1,p_2;p_1',p_2')a^+_{\alpha ip_1}a^+_{\beta jp_2}a_{\beta jp_2'}a_{\alpha ip_1'},$$

where $U_{\alpha i,\beta j}$ have properties analogous to (6) and (7).

For the creation operator $Q^+_{\gamma 0}$ of an exciton of the species γ, we have for all α and β

$$[H_{int}^{(\alpha\beta)}, Q_{T0}^+]=0, \quad Q_{T0}^+ = \frac{1}{\sqrt{N_0}} \sum_p a_{T2p}^+ a_{T1,-p}^+,$$

i.e., interactions of the excitons with particles of other species are absent. Therefore

$$[H_{int}, Q_{\alpha 0}^+]=[H_{int}^{(\alpha)}, Q_{\alpha 0}^+]=E_{\alpha 0} Q_{\alpha 0}^+,$$

and the state $\prod_\alpha (Q_{\alpha 0}^+)^{N_\alpha}|0\rangle$ with exciton condensate is an exact eigenstate of the system with eigenvalue $\sum_\alpha (\varepsilon_{\alpha 0}+E_{\alpha 0}) N_\alpha$.

This will be the ground state if the potentials $V_{\alpha\beta}(\mathbf{r})$ of the interactions of particles of different species are weaker than $U_\alpha(\mathbf{r})$. Otherwise, it may be energetically more advantageous for pairing of particles with $\alpha \neq \beta$.

Thus, the interaction of the excitons with the dynamical subsystem of the other particles is absent. The same is true with regard to the interaction with external static fields such that

$$V_1(\mathbf{r})=-V_2(\mathbf{r})\equiv V(\mathbf{r}). \tag{34}$$

Indeed, for the Hamiltonian of the interaction with a field

$$\hat{V} = \sum_i \sum_{p',p} V_i(p',p) a_{ip'}^+ a_{ip}$$

we obtain from (34) and the condition (9) on the wave functions of the particles the equation $V_1(p', p) = -V_2(-p, -p')$, from which there follows [23] $[\hat{V}, Q_0^+] = 0$. For example, for 2D magnetic excitons this means that their interaction with external fields of the form (34) is manifested only when allowance is made for virtual transitions to the higher Landau levels (or when the field V is included in the zeroth approximation, see Sec. 6.3). In the limit $H \to \infty$, it can be treated by perturbation theory and has smallness $\langle V\rangle/\omega_C \ll 1$, where $\langle V\rangle$ is the characteristic energy of the particles in the field V.

An exception is the case of fields in which the ground state of a "delocalized" zero-momentum exciton is unstable with respect to the formation of bound states (as, for example, in the field of a charged Coulomb center [23]) which, of course, cannot be eliminated in a finite order of perturbation theory.

Conclusions

We have found a class of exactly solvable many-particle two-component quantum models with arbitrary form of the two-body interaction for which it is possible to construct exact eigenstates corresponding to a condensate of noninteracting composite particles. The fulfillment of the necessary symmetry conditions between the components means effectively that there are no many-particle correlations in the state with condensate. This can be regarded as the reason why an exact solution is possible. In the case of physical realizations, for which the required properties cannot be satisfied with complete accuracy, our treatment may be helpful as a good initial approximation.

Examples of such systems are quasitwo-dimensional electron–hole (and multicomponent electron) systems in a strong magnetic field, when as a result of the action of the field and size quantization the particles have no kinetic energy. There could also be other physical realizations, including discrete models.

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